COHOMOLOGY OF CONES

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1. Summary

If \( X \subset \mathbb{P}^N \) is a smooth projective variety with dimension \( n \) and \( C(X) \) is the projective cone over \( X \), then if \( C(X) \) satisfies Poincare duality over \( \mathbb{Z} \) we must have \( H^k(X;\mathbb{Z}) \cong H^{k+2}(X;\mathbb{Z}) \) for all \( k \), and I think the multiplication by the class of a hyperplane gives the isomorphism. Similar statement for Poincare duality over \( \mathbb{Q} \), with \( \mathbb{Q} \)-coefficients. When \( X \) is a hypersurface of degree \( d > 1 \) this is impossible, as is shown by an explicit calculation of the cohomology of \( X \) (or at least all of its Betti numbers).

However, if \( d < N \), \( C(X) \) has terminal singularities and when \( N > 3 \) \( X \) is \( \mathbb{Q} \)-factorial. Not sure about analytically \( \mathbb{Q} \)-factorial but I would guess so (we are only dealing with one isolated singularity, and its a cone point...).

2. (Co)Homology of Cones

Let \( X \subset \mathbb{P}^N \) be a smooth projective variety and let \( C(X) \subset \mathbb{P}^{N+1} \) be the (projective) cone over \( X \). We begin with a basic observation:

**Proposition 1.** The projective cone \( C(X) \) is the Thom space of the geometric line bundle \( L \) on \( X \) associated to the invertible sheaf \( \mathcal{O}_X(1) \).

**Remark.** I am following “Fulton” conventions for moving between locally free sheaves and vector bundles. This means that \( \mathcal{O}_X(1) \) is the sheaf of local sections of \( L \). If this irritates you ... sorry. In particular, \( L \) has a global section.

**Proof.** Recall that the Thom space \( \text{Th}(L) \) can be constructed as follows: start with the \( \mathbb{P}^1 \)-bundle \( \mathbb{P}(\mathcal{O}_X(1) \oplus \mathcal{O}_X) \). It has 2 interesting global sections, \( \sigma_0, \sigma_\infty \) corresponding to the inclusions

\[
X \cong \mathbb{P}(\mathcal{O}_X) \subset \mathbb{P}(\mathcal{O}_X(1) \oplus \mathcal{O}_X) \text{ and } X \cong \mathbb{P}(\mathcal{O}_X(1)) \subset \mathbb{P}(\mathcal{O}_X(1) \oplus \mathcal{O}_X)
\]

The difference between these global sections is that the normal bundle of \( \sigma_0(X) \) can be identified with \( \mathcal{O}_X(1) \) while the normal bundle of \( \sigma_\infty(X) \) can be identified with \( \mathcal{O}_X(-1) \). We have

\[
\text{Th}(L) = \mathbb{P}(\mathcal{O}_X(1) \oplus \mathcal{O}_X)/\mathbb{P}(\mathcal{O}_X(1))
\]

(this may not be the most standard description, but see [Ati89]). To see that this is the cone, blow up the vertex \( p \in C(X) \) and observe that

- \( \text{Bl}_p C(X) \cong \mathbb{P}(\mathcal{O}_X(1) \oplus \mathcal{O}_X) \) and
- The exceptional divisor \( E \subset \text{Bl}_p C(X) \) over \( p \) is exactly \( \mathbb{P}(\mathcal{O}_X(1)) \).

This is just a projective version of the fact that the blowup of the affine cone \( C_{\text{aff}}(X) \) at the vertex \( p \subset C_{\text{aff}}(X) \) is the geometric line bundle \( L^v \) associated to \( \mathcal{O}_X(-1) \), with the exceptional divisor \( E \subset C_{\text{aff}}(X) \) corresponding to the zero-divisor \( X \subset L^v \).

\[\square\]

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Remark. Alternatively, view points \( l \in X \) as lines \( l \subset \mathbb{A}^{N+1} \). Then a vector in \( L_i \) is a linear functional \( \lambda : l \to \mathbb{C} \). The graph of \( \lambda \) is a line \( \lambda(l) \subset \mathbb{A}^{N+2} \), which we can view as a point \( \lambda(l) \in \mathbb{P}^{N+1} \). Since omitting the last coordinate of \( \lambda(l) \) gives back the line \( l \), we see that in fact \( \lambda(l) \subset C(X) \), and so we have a map

\[
\varphi : L \to C(X)
\]

At this point one checks that it’s an isomorphism onto \( C(X) \setminus \{p\} \), and as \( \lambda \to \infty, \lambda(l) \to p \), so that \( \varphi \) extends to the one-point-compactification \( Th(L) \), yielding a homeomorphism \( Th(L) \simeq C(X) \).

Now let’s recall the classic

**Theorem 1** (Thom isomorphism theorem). Let \( X \) be a reasonable space (say with the homotopy type of a CW complex) and let \( E \xrightarrow{\tau} X \) be an oriented real vector bundle. Then there is a class \( \tau(E) \in \check{H}^n(Th(E); \mathbb{Z}) \) generating \( \check{H}(Th(E); \mathbb{Z}) \) as a free \( H^*(X; \mathbb{Z}) \)-module of rank 1.

There is a parallel Thom isomorphism identifying \( H_i(X; \mathbb{Z}) \simeq \check{H}_{i+r}(Th(E); \mathbb{Z}) \).

**Remark.** The \( H^*(X; \mathbb{Z}) \)-module structure comes from the identifications \( H_i(X; \mathbb{Z}) \simeq \check{H}_{i+2}(C(X); \mathbb{Z}) \).

Applying this result, we obtain

**Proposition 2.** There is a class \( \tau(L) \in \check{H}^2(C(X); \mathbb{Z}) \) generating \( \check{H}^*(C(X); \mathbb{Z}) \) as a free \( H^*(X; \mathbb{Z}) \)-module of rank 1. Similarly, there are identifications \( H_i(X; \mathbb{Z}) \simeq \check{H}_{i+2}(C(X); \mathbb{Z}) \).

**Remark.** In the matter at hand, the tildes translate to:

\[
H^k(C(X); \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0 \\
0 & \text{if } k = 1 \\
H^{k-2}(X; \mathbb{Z}) & \text{if } k > 1
\end{cases}
\]

Now: assuming \( X \) is smooth, we have a fundamental class \( [X] \in H_{2n}(X; \mathbb{Z}) \) (here \( n \) is the complex dimension of \( X \)) and Poincare duality states that the cap product with the fundamental class

\[
H^k(X; \mathbb{Z}) \to H_{2n-k}(X; \mathbb{Z})
\]

sending \( \alpha \mapsto \alpha \cap [X] \)

is an isomorphism. We also have the universal coefficient formula, which provides exact sequences

\[
0 \to \text{Ext}^1(H_{k-1}(X; \mathbb{Z}), \mathbb{Z}) \to H^k(X; \mathbb{Z}) \to \text{Hom}(H_k(X; \mathbb{Z}), \mathbb{Z}) \to 0
\]

Of course, we can say much more about the general structure of \( H^*(X; \mathbb{Z}) \), using e.g. the hard Lefschetz theorem - more on that later.

Suppose for a minute that Poincare duality also holds on \( C(X) \). Which is to say, we have isomorphisms

\[
H^k(C(X); \mathbb{Z}) \simeq H_{2(n+1)-k}(C(X); \mathbb{Z})
\]

presumably given by capping with a fundamental class. Note that the obvious choice of fundamental class would be the image of \( [X] \) under the isomorphism \( H_{2n}(X; \mathbb{Z}) \simeq H_{2(n+1)}(C(X); \mathbb{Z}) \).

This will place serious restrictions on the (co)homology of \( X \), since we must have

\[
H^k(X; \mathbb{Z}) \simeq H^{k+2}(C(X); \mathbb{Z}) \simeq H_{2(n+1)-k-2}(C(X); \mathbb{Z}) \simeq H_{2n-k-2}(X; \mathbb{Z})
\]

Now Poincare duality on \( X \) provides an isomorphism

\[
H_{2n-k-2}(X; \mathbb{Z}) \simeq H^{k+2}(X; \mathbb{Z})
\]
and in this way we see that $H^k(X; \mathbb{Z}) \simeq H^{k+2}(X; \mathbb{Z})$ for all $k$. Also, it should be noted that since $H^1(C(X); \mathbb{Z}) = 0$ we must have $H_{2(n+1)-1}(C(X); \mathbb{Z}) = 0$ and hence $H_2(C(X); \mathbb{Z}) = 0$, and so $H^1(X; \mathbb{Z}) = 0$. Since $H^0(X; \mathbb{Z}) = \mathbb{Z}$ we conclude that

$$H^k(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Remark. I am pretty sure that the isomorphism $H_{2n-k-2}(X; \mathbb{Z}) \simeq H^{k+2}(X; \mathbb{Z})$ obtained above coincides with multiplication by the Chern class $c_1(\mathcal{O}_X(1))$. Given $\alpha \in H^k(X; \mathbb{Z})$, we obtain $\alpha \sim \tau \in H^{k+2}(C(X); \mathbb{Z})$. From this we obtain $\alpha \sim \tau \cap [C(X)] \in H_2(C(X); \mathbb{Z})$ and ... see here’s where I really need to know the homology version of the Thom isomorphism. (Idea: this is the pullback of $\tau$ along the usual inclusion $X \subset C(X)$). Knowing this would put even further restrictions on $X$.

The basic example of this phenomenon is when $X \subset \mathbb{P}^n$ is a linear subspace, hence so is $C(X) \subset \mathbb{P}^{n+1}$. It’s a little difficult to think of other such examples.

I’d like to also observe that our conditions on $H^r(X; \mathbb{Z})$ are not sufficient to guarantee Poincare duality for $H^r(C(X); \mathbb{Z})$. To see this, let $X \subset \mathbb{P}^2$ be a conic. Assuming the remark, Poincare duality for $C(X)$ would imply that multiplication by $c_1(\mathcal{O}_X(1))$ gives an isomorphism $\mathbb{Z} \simeq H_0(X; \mathbb{Z}) \simeq H^2(X; \mathbb{Z}) \simeq \mathbb{Z}$ which is false (it acts as multiplication by 2). Note however that if we worked over $\mathbb{Q}$ or a finite field $k$ of characteristic not 2 (instead of $\mathbb{Z}$, multiplication by $c_1$ actually would give an isomorphism.

The reason one should expect some funny business at the prime 2 in this example is that $C(X)$ is isomorphic to the quotient of $\mathbb{P}^2$ by the involution (a.k.a. $\mathbb{Z}/2$-action

$$t : \mathbb{P}^2 \to \mathbb{P}^2 \text{ sending } [x, y, z] \mapsto [-x, -y, z]$$

Similar remarks hold for rational normal curves of degree $d$, Veronese embeddings of $\mathbb{P}^n$, etc.

2.1. The singularity class of a cone point. I recall a simplified form of the criteria in Lemma 3.1 of Singularities of the MMP:

Proposition 3. Let $X \subset \mathbb{P}^N$ be a smooth projective variety. Then the projective cone $C(X) \subset \mathbb{P}^{N+1}$

is $\mathbb{Q}$-Gorenstein if and only if $r \cdot c_1(\mathcal{O}_X(1)) = K_X$ for some $r \in \mathbb{Q}$, and in this situation $C(X)$ is

- terminal if and only if $r < -1$,
- canonical if and only if $r \leq -1$,
- klt if and only if $r < 0$ and
- lc if and only if $r \leq 0$.

More precisely, if we resolve the singularities of $C(X)$ by blowing up the vertex, the discrepancy of the exceptional divisor $E \subset \text{Bl}_0 C(X)$ is $-1 - r$.

Some relevant corollaries, in no particular order:

Example 1. Suppose $X$ is a degree $d$ hypersurface. Then $\omega_X \simeq \mathcal{O}_X(d - N - 1)$, and so we have

$$r \cdot c_1(\mathcal{O}_X(1)) = K_X \text{ with } r = d - N - 1$$

Hence we see that $C(X)$ is terminal when $d < N$, canonical when $d = N$ and lc when $d = N + 1$. When $d > N + 1$ it’s not even lc.

One can generalize this example to complete intersections.

Example 2. More generally, a cone over an anti-canonically embedded Fano varieity is always at least klt. A cone over a variety with trivial canonical (e.g. a Calabi-Yau variety) is always at least lc.
2.2. The link at a cone point. Looking into any of the standard proofs of Poincare duality one sees that a key property of a manifold \( M \) exploited at various stages is that for any point \( p \in M \),

\[
H^k(M, M \setminus \{p\}; \mathbb{Z}) \simeq \begin{cases} 
\mathbb{Z} & \text{if } k = \dim M \\
0 & \text{otherwise}
\end{cases}
\]

This property is axiomatized as follows: let \( X \) be a reasonable topological space (e.g. a CW-complex).

**Definition 1.** \( X \) is a homology \( n \)-manifold if and only if for every point \( p \in X \),

\[
H^k(X, X \setminus \{p\}; \mathbb{Z}) \simeq \begin{cases} 
\mathbb{Z} & \text{if } k = n \\
0 & \text{otherwise}
\end{cases}
\]

If \( Y \) is an \( n \)-dimensional complex variety, then it is generically smooth, so it could only be a homology \( 2n \)-manifold. Furthermore if \( p \in Y \) is a point with a neighborhood \( U \subset X \) that deformation-retracts onto \( p \), then by excision \( H^k(Y, Y \setminus \{p\}; \mathbb{Z}) \simeq H^k(U, U \setminus \{p\}; \mathbb{Z}) \) and by the relative cohomology exact sequence \( H^k(U, U \setminus \{p\}; \mathbb{Z}) \simeq H^{k-1}(U \setminus \{p\}; \mathbb{Z}) \). If \( Y \) is an affine variety sitting in \( \mathbb{C}^N \) (always the case locally) and \( \mathcal{S}_e(p) \) is a sphere of radius \( e \) centered at \( p \), then for suitably small \( U \) and \( e \) one has \( U \setminus \{p\} \approx \mathcal{S}_e(p) \) where here \( \approx \) denotes homotopy equivalence. In this way we see that

\[
H^k(Y, Y \setminus \{p\}; \mathbb{Z}) \simeq H^{k-1}(\mathcal{S}_e(p); \mathbb{Z}) \text{ for all } k
\]

**Definition 2.** The space \( \mathcal{S}_e(p) \) is called the link of \( X \) at \( p \).

To justify the terminology “the” one shows that it is independent of \( e \) for sufficiently small \( e \) (up to homeomorphism, say).

**Proposition 4.** If \( X \subset \mathbb{P}^N \) is a smooth projective variety and \( \mathcal{C} \subset \mathbb{P}^{N+1} \) is the affine cone over \( X \), with vertex \( p \in C(X) \), then the link \( \mathcal{S}_e(p) \) is the \( S^1 \)-bundle (a.k.a. circle bundle) associated to the invertible sheaf \( \mathcal{O}_X(-1) \).

**Proof.** Let \( \pi : \text{Bl}_p C_a(X) \rightarrow C(X) \) be the blow-up of \( C_a(X) \) at \( p \). Recall that \( \text{Bl}_p C_a(X) \simeq L^\vee \), the geometric line bundle associated to \( \mathcal{O}_X(-1) \), with exceptional divisor \( E \simeq X \) corresponding to the 0-section. The preimage of a \( \epsilon \)-sphere \( \mathcal{S}_e(p) \subset C_a(X) \) at \( p \) is the \( \epsilon \)-sphere bundle of \( L^\vee \).

To relate the topology of \( \mathcal{S}_e(p) \) to that of \( X \), we can use the long exact sequence on homotopy groups

\[
\cdots \rightarrow \pi_i(S^1) \rightarrow \pi_i(\mathcal{S}_e(p)) \rightarrow \pi_i(X) \xrightarrow{\partial} \pi_{i-1}(S^1) \rightarrow \cdots
\]

Since \( \pi_i(S^1) = 0 \) for \( i > 1 \) and all the spaces are connected, this reduces to an exact sequence

\[
0 \rightarrow \pi_2(\mathcal{S}_e(p)) \rightarrow \pi_2(X) \rightarrow \mathbb{Z}
\]

\[
\rightarrow \pi_1(\mathcal{S}_e(p)) \rightarrow \pi_1(X) \rightarrow \pi_0(S^1) \rightarrow 0
\]

together with isomorphisms \( \pi_i(\mathcal{S}_e(p)) \simeq \pi_i(X) \) for \( i > 2 \). As for cohomology, we have a Gysin sequence of the form

\[
\cdots \rightarrow H^{k-2}(X; \mathbb{Z}) \xrightarrow{-c_1} H^k(X; \mathbb{Z}) \xrightarrow{\pi} H^k(\mathcal{S}_e(p); \mathbb{Z})
\]

\[
\rightarrow H^{k-1}(X; \mathbb{Z}) \rightarrow \cdots
\]

where \( c_1 \) is the first Chern class of \( \mathcal{O}_X(1) \) and \( \pi : \mathcal{S}_e(p) \rightarrow X \) is the projection.

Now let’s recall a variant of the hard Lefschetz theorem:
**Theorem 2** (Lefschetz). Let \( X \) be a smooth projective variety of dimension \( n \) and let \( c_1 \) be its first Chern class. Then multiplication by \( c_1 \)

\[
H^k(X; \mathbb{Q}) \to H^{k+2}(X; \mathbb{Q})
\]

is injective for \( k < n \), and surjective for \( k > n \).

**Remark.** This is only true with \( \mathbb{Q} \) coefficients, as one can see by considering a rational normal curve of degree \( d > 1 \) (or more generally a Veronese embedding of degree \( d > 1 \)). However via the universal coefficient theorem one obtains a statement about integral cohomology (below the middle dimension the kernel of \( c_1 \) is torsion, above the middle dimension the cokernel is torsion).

**Remark.** It’s because of this theorem that the Hodge diamond is, well, a diamond.

Applying this theorem we see that after tensoring with \( \mathbb{Q} \), for \( k - 2 < n \) the Gysin sequence breaks up into short exact sequence

\[
0 \to H^{k-2}(X; \mathbb{Q}) \xrightarrow{c_1} H^k(X; \mathbb{Q}) \to H^k(S_c(p); \mathbb{Q}) \to 0
\]

Similarly for \( k - 2 > n \) we have short exact sequences

\[
0 \to H^{k-1}(S_c(p); \mathbb{Q}) \to H^{k-2}(X; \mathbb{Q}) \to H^k(X; \mathbb{Q}) \to 0
\]

**Example 3.** Let’s actually take a closer look at cone over a Veronese. Let \( X \subset \mathbb{P}^n \) be the image of \( \mathbb{P}^n \) under the \( d \)-th Veronese embedding, and let \( C(X) \) be the cone over \( X \), with vertex \( p \). Then \( \mathcal{O}_X(1) \cong \mathcal{O}_{\mathbb{P}^n}(d) \) and so \( c_1(\mathcal{O}_X(d)) = dh \), where \( h = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \). Hence the Gysin exact sequence looks like

\[
\cdots \to H^{k-2}(\mathbb{P}^n; \mathbb{Z}) \xrightarrow{-dh} H^k(\mathbb{P}^n; \mathbb{Z}) \xrightarrow{\pi^*} H^k(S_c(p); \mathbb{Z}) \xrightarrow{c_1} H^{k+2}(\mathbb{P}^n; \mathbb{Z}) \to \cdots
\]

Since \( H^k(\mathbb{P}^n; \mathbb{Z}) = \mathbb{Z} \) generated by \( h^k \) if \( k \) is even and 0 otherwise, and since multiplication by \( -dh \) is always injective, we see that \( H^k(S_c(p); \mathbb{Z}) = 0 \) for \( k \) odd and we obtain short exact sequences

\[
0 \to \mathbb{Z} \xrightarrow{-d} \mathbb{Z} \to H^k(S_c(p); \mathbb{Z}) \to 0
\]

for \( k \) even, showing that \( H^k(S_c(p); \mathbb{Z}) \cong \mathbb{Z}/d \) for even \( k \). This is not surprising since the description of \( C(X) \) as a quotient of \( \mathbb{P}^{n+1} \) by an action of \( \mu_d \) (if \( \zeta \in \mu_d \) is a primitive root, then it acts on \( [x_0, \ldots, x_{n+1}] \) like

\[
\zeta \cdot [x_0, \ldots, x_{n+1}] = [\zeta x_0, \ldots, \zeta x_n, x_{n+1}] ;
\]

the fixed point \( [0, \ldots, 0, 1] \) corresponds to the cone point) identifies \( S_c(p) \) with a lens space obtained as the quotient of a free action of \( \mu_d \) on \( S^{2n+1} \).

2.3. **Singular cohomology of hypersurfaces.** To see how the above discussion plays out in some specific cases it will be nice to know the singular cohomology of smooth hypersurfaces (and more generally complete intersections). I actually don’t know a reference for the ensuing calculations so I will just go for it.

Let \( X \subset \mathbb{P}^{n+1} \) be a smooth hypersurface and let \( i : X \to \mathbb{P}^{n+1} \) be the inclusion. Recall

**Theorem 3** (Lefschetz). The restriction map \( i^* H^k(\mathbb{P}^{n+1}; \mathbb{Z}) \to H^k(X; \mathbb{Z}) \) is injective for \( k \leq n \) and an isomorphism for \( k < n \).

Knowledge of the cohomology of \( \mathbb{P}^{n+1} \) shows that for \( k < n \)

\[
H^k(X; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{if } k \text{ is odd} \\
0 & \text{otherwise}
\end{cases}
\]
For simplicity I will assume \( n \) is (the case where \( n \) is odd is slightly more complicated). In that case we have an injection \( \mathbb{Z} \to H^n(X; \mathbb{Z}) \). Poincare duality together with the universal coefficient theorem then shows that \( H^k(X; \mathbb{Z}) \) is torsion-free for all \( k \) and for \( k > n \),

\[
H^k(X; \mathbb{Z}) \simeq \begin{cases} 
\mathbb{Z} & \text{if } k \text{ is odd} \\
0 & \text{otherwise}
\end{cases}
\]

The only thing left to do is compute the rank of \( H^n(X; \mathbb{Z}) \) (of course one might also want to know about the intersection form - maybe another day). The preceding discussion shows

\[
\chi(X) = \sum_k \text{rk} H^k(X; \mathbb{Z}) = n + \text{rk} H^n(X; \mathbb{Z})
\]

and so we just need to calculate \( \chi(X) \). For this we can use the formula

\[
\chi(X) = \int_X c_n(\tau_X)
\]

the integral of the top Chern class of the tangent bundle. To get going on this integral, note that there is a short exact sequence of vector bundles on \( X \)

\[
0 \to \tau_X \to i^* \tau_{\mathbb{P}^{n+1}} \to \mathcal{N}_{X|\mathbb{P}^{n+1}} \to 0
\]

and hence

\[
c(\tau_X) = \frac{i^* c(\tau_{\mathbb{P}^{n+1}})}{c(\mathcal{N}_{X|\mathbb{P}^{n+1}})}
\]

From the Euler exact sequence on \( \mathbb{P}^{n+1} \) we find that

\[
c(\tau_{\mathbb{P}^{n+1}} = c(\mathcal{O}_{\mathbb{P}^{n+1}(1)})^{n+2} = (1 + h)^{n+2}
\]

and since \( \mathcal{N}_{X|\mathbb{P}^{n+1}} \simeq \mathcal{O}_X(d) \) where \( d = \text{deg} X \), we compute

\[
c(\tau_X) = \frac{(1 + h)^{n+2}}{1 + dh}
\]

(where I am abusively dropping the \( i^* \) in \( i^* h \)). We need to expand this as a power series in \( h \):

\[
\frac{(1 + h)^{n+2}}{1 + dh} = \left( \sum_j (-1)^j d^j h^j \right) \cdot \left( \sum_k \binom{n + 2}{k} h^k \right)
\]

\[
= \sum_{j,k} (-1)^j \binom{n + 2}{k} h^{j+k}
\]

and now recall that the integral will only pick off the degree \( n \) term: so, we find

\[
\chi(X) = \sum_{j+k=n} (-1)^j \binom{n + 2}{k} \int_X h^n
\]

and since \( \int_X h^n = d \) this is just

\[
\sum_{j+k=n} (-1)^j \binom{n + 2}{k} = \sum_{k=0}^n (-1)^{n-k} d^{n-k+1} \binom{n + 2}{k}
\]

\[
= \frac{1}{d} \left( (1 - d)^{n+2} + (n + 2)d - 1 \right)
\]

after a little bit of rearranging. Combining this with the formula \( \chi(X) = n + \text{rk} H^n(X; \mathbb{Z}) \) we obtain

\[
\text{rk} H^n(X; \mathbb{Z}) = \frac{1}{d} \left( (d - 1)^{n+2} + (n + 2)d - 1 \right) - n
\]
If $n$ is odd, the Chern class calculation is identical, but we have $\chi(X) = n + 1 - \text{rk}H^n(X; \mathbb{Z})$, and so

$$
\text{rk}H^n(X; \mathbb{Z}) = n + 1 - \frac{1}{d}((1 - d)^{n+2} + (n + 2)d - 1)
$$

as a reality check, note that when $n = 1$ we recover the classic formula for the genus $g$ of a plane curve $X$ in terms of its degree: for in that situation

$$
2g = \text{rk}H^1(X; \mathbb{Z}) = \frac{(d - 1)^3 + 1}{d} - 1
$$

so that $g = \frac{(d-1)(d-2)}{2}$. Lovely! Note also that all the formulas for the rank output 1 when $d = 1$ (so $X = \mathbb{P}^n$), as they must.

References