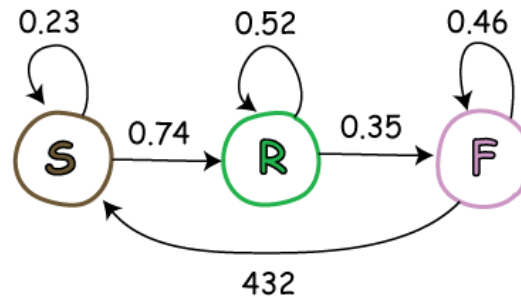


1. Spotted knapweed is an invasive species of noxious weed. They are a big problem in Canada.



Their lifecycle consists of three stages: seed, rosette, flower and can be modeled as follows:

1. Seed does not germinate but remains viable 0.23
2. Seed germinates into a rosette 0.74
3. Rosette remains rosette 0.52
4. Rosettes do not produce seeds
5. Rosette blooms into flowers 0.35
6. Flower remains flower 0.46
7. Flowers produce on average 432 seeds



As with the hippos, this population can be modeled with a matrix of the form:

$$K = \begin{bmatrix} p_{1,1} & b_2 & b_3 \\ p_{1,2} & p_{2,2} & 0 \\ 0 & p_{2,3} & p_{3,3} \end{bmatrix}$$

Where  $b_i$  is the average amount of new seeds produced in the  $i$ th life stage, and  $p_{i,j}$  is the probability something in stage  $i$  transitions to stage  $j$ .

*Note: This matrix is referred to as the Lefkovich matrix or, sometimes, as the Leslie matrix.*

- (a) Say the population after one generation is  $\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$  for the general matrix  $K$ , write out the product  $K\vec{s}$ .

(b) Using your previous answer, explain why  $K\vec{s}$  models the amount of each life stage type in the next generation.

(c) Fill in the matrix  $K$  to model the spotted knapweed.

(d) Write the characteristic polynomial of  $K$ . You do not need to simplify.

(e) This has only one eigenvalue. It is

$$\lambda = \frac{121}{300} + \frac{1}{4050000} \left( (3716328850998750000000 - 22143375000000\sqrt{28167003253545333})^{1/3} \right. \\ \left. + 13500(1510472530 + 9\sqrt{28167003253545333})^{1/3} \right) \\ \approx 5.22.$$

Explain the meaning of 5.22 for the spotted knapweed population.

(f) Write out a matrix that has the same nullspace as the  $\lambda$ -eigenspace. (You can either use  $\lambda$  or 5.22 in your answer.)

(g) A compute solver gives an eigenvector,  $\vec{u} = \begin{bmatrix} .86 \\ .13 \\ .01 \end{bmatrix}$ . Explain the meaning of this vector in terms of the spotted knapweed population.

(h) How would you recommend combatting the spotted knapweed?

2. We consider the matrix  $A = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$ .

(a) Find the eigenvalues of  $A$ .

(b) Do we have enough information to determine that  $A$  is diagonalizable? Why or why not?

(c) Determine bases for the eigenspace(s) of  $A$ .

(d) Find matrices  $P$ ,  $P^{-1}$ , and  $D$  such that  $A = PDP^{-1}$ . Verify that your decomposition is correct.

(e) Use your diagonalization to calculate  $e^A$ . Present your answer as a single  $2 \times 2$  matrix.

3. Chris and Matt are big pizza eaters, and also big on knowing their macro nutrients. Both agree on the same kind of pizza dough, but prefer different toppings. Matt likes feta cheese and sunflower seeds.

Chris prefers egg and kale. These have macro profiles  $\begin{bmatrix} \text{Fat} \\ \text{Carbs} \\ \text{Protein} \end{bmatrix} \in \mathbb{R}^3$  as follows:

$$\vec{d} = \begin{bmatrix} 3 \\ 42 \\ 7 \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} 21 \\ 4 \\ 14 \end{bmatrix}, \quad \vec{s} = \begin{bmatrix} 50 \\ 24 \\ 19 \end{bmatrix}, \quad \vec{e} = \begin{bmatrix} 10 \\ 1 \\ 13 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}.$$

Let  $\mathcal{M} = \{\vec{d}, \vec{f}, \vec{s}\}$  and  $\mathcal{C} = \{\vec{e}, \vec{k}\}$ . Also, let  $M = [\vec{d} \ \vec{f} \ \vec{s}]$ , and  $C = [\vec{e} \ \vec{k}]$ . These have inverses:

$$M^{-1} = \frac{1}{13990} \begin{pmatrix} -260 & 301 & 304 \\ -630 & -293 & 2028 \\ 560 & 105 & -870 \end{pmatrix}, \quad C^{-1} = \frac{1}{306} \begin{pmatrix} 179 & 16 & -129 \\ 35 & -2 & -3 \\ -581 & -28 & 417 \end{pmatrix}$$

(a) A friend offers Matt some carrot cake with profile  $\vec{c} = \begin{bmatrix} 20 \\ 45 \\ 4 \end{bmatrix}$ . Matt passes and says he would prefer to make a pizza. What quantities of each of his preferred ingredients should he use? It is okay to express your answer just in terms of the symbols for the matrices.

(b) Chris opts to have 100g of carrot cake, but wants to log the macros in terms of his fav pizza ingredients. Write an expression that describes this.

(c) Write a matrix and vector expression relating  $\vec{c}_C$  and  $\vec{c}_M$ .

(d) What are the entries  $f_1, f_2, f_3$  of  $\vec{f}_M = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}_M$  ?

(e) Simplify the expression  $M\vec{f}_M - C^{-1}\vec{f} - C\vec{f}_C - M^{-1}C\vec{f}_C$ .