1. Recall that a subset S of $\mathbb{R}^n$ is a subspace if i) $\vec{0} \in S$, ii) $\vec{u}+\vec{v} \in S$ for all $\vec{u}, \vec{v} \in S$, and iii) $c\vec{u} \in S$ for all $c \in \mathbb{R}, \vec{u} \in S$.

   Determine (by checking the properties, drawing a picture and/or using some theorem) whether or not the following sets describe subspaces:

   - the subset of $\mathbb{R}^2$ consisting of vectors of the form $\begin{bmatrix} a \\ b \end{bmatrix}$, where $-a = 1 + 3b$.

     Solution: No. $\vec{0}$ isn't in there.

   - the subset of $\mathbb{R}^3$ consisting of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $bc = 0$

     Solution: No. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ isn't in there, but each summand is.

   - the subset of $\mathbb{R}^4$ consisting of vectors of the form $\begin{bmatrix} 5a \\ 2a+b \\ 3a+b-c \\ 2b-c \end{bmatrix}$

     Solution: Yes. This is the span of $\left\{ \begin{bmatrix} 5 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$.

   - the subset of $\mathbb{R}^3$ consisting of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $4a = 3b - c$.

     Solution: Yes. This is $\begin{bmatrix} \frac{1}{4}(3b-c) \\ b \\ c \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 3/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/4 \\ 0 \\ 1 \end{bmatrix} \right\}$.

   - the subset of $\mathbb{R}^2$ consisting of vectors of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ where $a + b \geq 0$

     Solution: No. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in there, but $-1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ isn't.

2. Let $A = \begin{bmatrix} 3 & 6 & 0 & 1 \\ 2 & 4 & 3 & -4 \\ -1 & -2 & -6 & 9 \end{bmatrix}$. Note that A has reduced echelon form:

$$B = \begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{14}{9} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and A^T has reduced echelon form

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis for the row space of A.

Solution: The nonzero rows of B form a basis for the rowspace of A: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{14}{9} \end{bmatrix} \right\}$.

(b) Find a different basis for the row space of A such that at least one vector in this new basis is **not** a multiple of a vector in your answer to (a).

Solution: The nonzero pivots of C tell us that the first and second rows of A form a basis for the rowspace of A: $\begin{bmatrix} 3 \\ 6 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ -4 \end{bmatrix}$.

(c) Find a basis for the column space of A.

Solution: The nonzero rows of C form a basis for the column space of A: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

(d) Find a different basis for the column space of A such that at least one vector in this new basis is **not** a multiple of a vector in your answer to (c).

Solution: The pivots of B tell us that the first and third column vectors of A form a basis for the column space of A: $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -6 \end{bmatrix}$.

(e) Is $\begin{bmatrix} 3 \\ 3 \\ 5 \\ -7 \end{bmatrix}$ in the row space? Is it in the column space?

Solution: It is not in the row space. To see this we setup the matrix equation

$$\left(\begin{array}{cc|c} 3 & 2 & 3 \\ 6 & 4 & 3 \\ 0 & 3 & 5 \\ 1 & -4 & -7 \end{array}\right)$$

A couple steps of row reduction leads to:

$$\left(\begin{array}{cc|c} 1 & \frac{2}{3} & 1 \\ 0 & 0 & -3 \\ 0 & 3 & 5 \\ 0 & -\frac{14}{3} & -8 \end{array}\right).$$

This is inconsistent. Since every vector in the row space is a linear combination of the basis vectors, this says $\begin{bmatrix} 3 \\ 3 \\ 5 \\ -7 \end{bmatrix}$ is not in the row space. It also is not in the column space, since it is the wrong dimension.

(f) Is the linear transformation $T(\vec{x}) = A\vec{x}$ injective, surjective or neither? Why?

Solution: Because the column space has 2 vectors, that the rank of A is 2. The nullity is also $4 - 2 = 2$. This means the range of T has dimension 2. Thus, $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ is not onto. T also is not one to one, because the kernel of T has dimension 2.

(g) Without doing any further calculations, find the nullity of A^T.

Solution: A^T has the same rank as A, which is 2. Notice that A^T is a 4×3 matrix. The rank theorem says that $2 + \text{nullity}(A^T) = 3$, thus the nullity is 1.

3. Violet is a chemical engineer for Starbucks and is attempting to design a new drink using nonfat milk, sugar, chemical X and compound Y. These have macro-nutrient profiles

$$\begin{bmatrix} \text{Calories} \\ \text{Fat} \\ \text{Carbs} \\ \text{Protein} \end{bmatrix} \in \mathbb{R}^4.$$

per 100g as follows:

$$\begin{bmatrix} M & S & X & Y \\ 32 & 400 & 895 & 575 \\ 0 & 0 & 99 & 99 \\ 5 & 100 & 0 & -50 \\ 3 & 0 & 1 & -29 \end{bmatrix}$$

This has reduced echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, when augmenting an arbitrary vector and row reducing, Violet finds that:

$$\left(\begin{array}{cccc|c} 1 & \frac{25}{2} & \frac{895}{32} & \frac{575}{32} & \frac{1}{32}K \\ 0 & 1 & -\frac{179}{48} & -\frac{179}{48} & \frac{2}{75}C - \frac{1}{240}K \\ 0 & 0 & 1 & 1 & \frac{1}{99}F \\ 0 & 0 & 0 & 0 & C + \frac{9}{4}F - \frac{1}{4}K + P \end{array}\right)$$

(a) Describe the set, S, of possible drinks that Violet could make. (We will assume that negative amounts of an ingredient are possible. Say by telling customers to do very specific exercises in combination with the drinks.)

> **Solution:** All profiles that satisfy: $C + 9F - K + P = 0$.

(b) Show that S is a subspace by confirming each of the the three properties.

> **Solution:** $\vec{0}$ is in S.
>
> If we take two elements $\vec{u} = \begin{bmatrix} K \\ F \\ C \\ P \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} K' \\ F' \\ C' \\ P' \end{bmatrix}$, then we see that $\vec{u} + \vec{v} = \begin{bmatrix} K + K' \\ F + F' \\ C + C' \\ P + P' \end{bmatrix}$ also belongs to S since
>
> $(C + C') + 9(F + F') - (K + K') + (P + P') = C + 9F - K + P + C' + 9F' - K' + P' = 0 + 0 = 0.$
>
> Lastly, if $\vec{u} \in S$ and $r \in R$ then we have
>
> $$rC + 9rF - Kr + Pr = r(C + 9F - K + P) = r0 = 0.$$
>
> So, $r\vec{u}$ also belongs to S.

(c) List all collections of ingredients that could form a basis for the subspace.

> **Solution:** $\{\vec{m}, \vec{s}, \vec{X}\}$, $\{\vec{m}, \vec{s}, \vec{Y}\}$, $\{\vec{s}, \vec{X}, \vec{Y}\}$

(d) What is the dimension of the column space?

> **Solution:** 3

(e) Write a basis for the nullspace of the matrix containing the 4 column vectors.

> **Solution:** $\left\{ \begin{bmatrix} 10 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

(f) What is the nullity of the matrix containing the 4 column vectors?

> **Solution:** 1

(g) Violet decides to make a smoothie with profile, $\begin{bmatrix} 337 \\ 4 \\ 75 \\ 1 \end{bmatrix}$. Using one of the bases from the previous question describe the recipe to make this. You do not need to simplify anything.

> **Solution:** Let's use the basis $\{\vec{m}, \vec{s}, \vec{X}\}$. We see that this gives the matrix:
>
> $$\left(\begin{array}{ccc|r} 1 & \frac{25}{2} & \frac{895}{32} & \frac{575}{32}, K \\ 0 & 1 & -\frac{179}{48} & \frac{2}{75}\,C - \frac{1}{240}\,K \\ 0 & 0 & 1 & \frac{1}{99}\,F \\ 0 & 0 & 0 & C + \frac{9}{4}\,F - \frac{1}{4}\,K + P \end{array}\right)$$

So, we need $\frac{1}{99}F$ of $\vec{X}$. We need

$$\frac{179}{48}\vec{X} + \frac{2}{75}C - \frac{1}{240}K = \frac{1}{99}F + \frac{2}{75}C - \frac{1}{240}K$$

of $\vec{s}$, and

$$-\frac{25}{2}\vec{s} - \frac{895}{32}\vec{X} + \frac{575}{32}K = -\frac{25}{2}(\frac{1}{99}F + \frac{2}{75}C - \frac{1}{240}K) - \frac{895}{32}(\frac{1}{99}F) + \frac{575}{32}K$$

of $\vec{m}$.

The recipe is then evaluating $K = 337, F = 4, C = 75, P = 1$ in the vector:

$$\begin{bmatrix} -\frac{25}{2}(\frac{1}{99}F + \frac{2}{75}C - \frac{1}{240}K) - \frac{895}{32}(\frac{1}{99}F) + \frac{575}{32}K \\ \frac{1}{99}F + \frac{2}{75}C - \frac{1}{240}K \\ \frac{1}{99}F \end{bmatrix}$$

(h) Is there any other recipe that makes the same nutrition outcome *and* uses the same ingredients as the previous part? Why?

Solution: No. Representations in a basis are unique.