Math 308: Midterm 3 Review

Chris Fowler, Matthew Junge

May 13, 2016

Contents

1 Bases and Dimension

1.1 Subspaces

1.1.1 Checking if a set is a subspace

1. Decide if the following sets in $\mathbb{R}^n$ are subspaces or not. If it is justify by verifying the three properties or by showing it is equivalent to a set we already know is a subspace (span of vectors, range of a linear transformation, kernel, nullspace, etc...).

   (a) The set of all vectors $\vec{x} \in \mathbb{R}^3$ such that $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   Solution: No. $\vec{0}$ isn't in there.

   (b) The set of solutions in $\mathbb{R}^2$ to $y = x$.

   Solution: Yes. This is the set of vectors $\begin{bmatrix} x \\ x \end{bmatrix}$ which is the same as $x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So this is the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (c) The set of vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} a \\ b \\ a^2 \\ c \end{bmatrix}$.

   Solution: No. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in there, but $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is not.

   (d) The set of vectors in $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ of the form $\begin{bmatrix} 3(a+1) \\ b-1 \\ a-1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

   Solution: Yes. This is the same as span $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

   (e) The set of vectors $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^2$.

Solution: No. $\begin{bmatrix}1\\0\end{bmatrix}$ is in there but $2\begin{bmatrix}1\\0\end{bmatrix}$ is not.

1.1.2 Nullspace and Kernel

2. Let $A = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 1 & 0 & 1\end{bmatrix}$. Describe all vectors in the nullspace of A. Explain why the set you found is a subspace (without just saying that nullspaces are subspaces).

Solution: Well we can put A in reduced echelon form:

$$\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{bmatrix}$$

This says that $\begin{bmatrix}-x_4\\-x_4\\x_4\\x_4\end{bmatrix} = x_4\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$ is the nullspace.

This is a subspace because $x_4 = 0$ gives $\vec{0}$ is in there. The sum of two different vectors in the space corresponds to choosing two vectors

$$x_4\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} + x_4'\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} = (x_4 + x_4')\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}.$$

This is also in the space. Similarly, $rx_4\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$ belongs to the space for all $r \in \mathbb{R}$.

3. Let $T(\vec{x}) = \begin{bmatrix}1 & 0 & \frac{14}{11}\\0 & 1 & \frac{-19}{11}\\0 & 0 & 0\end{bmatrix}\vec{x}$.

Describe all vectors in the kernel of T.

Solution: It is the set of vectors of the form $\begin{bmatrix} -\frac{14}{11}t \\ \frac{19}{11}t \\ t \end{bmatrix}$.

4. Let $T(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 & -3 & 4 \\ 0 & 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

   Describe all vectors in the Kernel of T.

Solution: It is the set $x_5 \begin{bmatrix} -3 \\ 2 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -4 \\ 2 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

5. Consider the matrix A and it's transpose A^T.

$$A = \begin{pmatrix} 3 & -4 & -1 & 0 & -4 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \\ -1 & 1 & 1 & 2 & 2 & 0 \end{pmatrix} \xrightarrow{echelon} \begin{pmatrix} 1 & 0 & 0 & -2 & 2 & 0 \\ 0 & 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 3 & 1 & 0 & -1 \\ -4 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -4 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{echelon} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   Find all of these quantities in **TWO** different ways:

   - basis for col(A).

Solution:

$$\left\{\begin{bmatrix}3\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}-4\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\1\end{bmatrix}\right\} \text{ or } \left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}.$$

- basis for row(A).

Solution:

$$\left\{\begin{bmatrix}1\\0\\0\\-2\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-2\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\2\\2\\0\end{bmatrix}\right\} \text{ or } \left\{\begin{bmatrix}3\\-4\\-1\\0\\-4\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\2\\2\\0\end{bmatrix}\right\}.$$

- basis for col(A^T).

Solution: Same as row(A).

- basis for row(A^T).

Solution: Same as col(A).

- rank and nullity of A.

Solution: A has a rank of 3 and nullity of 3.

- rank and nullity of A^T.

Solution: A^T has a rank of 3 and a nullity of 1.

6. Expand the set $\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$ to be a basis for the subspace $w+x+y+z=0$.

Solution: We first observe that this is a subspace in $\mathbb{R}^4$ of dimension 3, so if we can find a set of three linearly independent vectors in this subspace then they form a basis. We notice that the vector $\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$ is also in this space. To check if all three are linearly independent we reduce the matrix

$$\begin{bmatrix}-1 & 0 & 1\\0 & -1 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix} \sim \begin{bmatrix}-1 & 0 & 1\\0 & -1 & 1\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}.$$

The echelon form matrix has three pivot columns so the vectors are linearly independent and our basis is

$$\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\right\}.$$

7. Find a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^4$ such that the kernel of T is PRECISELY the span of $\begin{bmatrix}1\\1\end{bmatrix}$.

Solution: We will do so by finding some matrix A and setting $T(\vec{x}) = A\vec{x}$. So A has to be 3×2 and $\text{nullity}(A) = \dim(\ker(T)) = 1$, meaning that A has rank 1 as well. So really the criterion is that A is nonzero and $A\begin{bmatrix}1\\1\end{bmatrix} = \vec{0}$. One such matrix would be

$$A = \begin{bmatrix}1 & -1\\0 & 0\\0 & 0\end{bmatrix}.$$

8. In this question we consider the vectors $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a) Explain why $\{\vec{u}, \vec{v}\}$ is a basis for $\mathbb{R}^2$.

Solution: These two vectors are clearly not scalar multiples of each other, so they are linearly independent and therefore form a basis for $\mathbb{R}^2$.

(b) Write any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in terms of this basis. That is, find c_1 and c_2 such the $\vec{x} = c_1\vec{u} + c_2\vec{v}$.

Solution: We set up the equation

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solving this system tells us that $c_2 = x_2 - x_1$ and $c_1 = 2x_1 - x_2$. So $\vec{x} = (2x_1 - x_2)\vec{u} + (x_2 - x_1)\vec{v}$.

(c) Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with $T(\vec{u}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $T(\vec{v}) = \begin{bmatrix} 0 \\ 10 \\ 6 \end{bmatrix}$. Use your answer in part (b) to write a formula for $T(\vec{x})$ and find the matrix A such that $T(\vec{x}) = A\vec{x}$.

Solution: For any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, from part (b) we know that

$$\begin{aligned} T(\vec{x}) &= T((2x_1 - x_2)\vec{u} + (x_2 - x_1)\vec{v}) \\ &= (2x_1 - x_2)T(\vec{u}) + (x_2 - x_1)T(\vec{v}) \\ &= (2x_1 - x_2)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (x_2 - x_1)\begin{bmatrix} 0 \\ 10 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 - x_2 \\ -6x_1 + 8x_2 \\ 3x_2 \end{bmatrix}. \end{aligned}$$

This also means $A = \begin{bmatrix} 2 & -1 \\ -6 & 8 \\ 0 & 3 \end{bmatrix}$.

9. Suppose that $T\colon \mathbb{R}^{100} \to \mathbb{R}^{52}$ with $T(\vec{x}) = A\vec{x}$. Also suppose that $S\colon \mathbb{R}^{52} \to \mathbb{R}^{19}$ with $S(\vec{x}) = B\vec{x}$.

   (a) Suppose that $\text{Dim}(\text{Ker}(T)) = 50$ what is the Nullity of A?

   Solution: The nullspace of A and kernel of T are the same, so $\text{nullity}(A) = 50$.

   (b) Suppose that B^T has nullity 12 what is the dimension of the range of S?

   Solution: B^T is a 52×19 matrix, so if it has nullity 12 then it has rank 7, as does B itself. Since the range of S is equivalent to the column space of B, its dimension must be the rank of B, or 7.

   (c) Do we know the nullity of BA?

   Solution: No, we can think of the rank of BA as the dimension of the range of $S \circ T$. But we have no way of knowing the overlap between the range of T and the kernel of S, so we do not have enough information to solve this problem.

   (d) How about if we knew that $\text{Rank}(BA)$ is 4?

   Solution: Then yes, BA is a 19×100 matrix, so if the rank is 4 then we know that the nullity must be $100 - 4 = 96$.