MATH 581G: HOMEWORK ASSIGNMENT # 3

DUE MONDAY, NOVEMBER 13

Problems 5.2 and 5.3 from Osserman's notes and the following:

(1) Let $\mathcal{O}_K$ be a Dedekind domain with field of fractions K. Let L and L' be finite separable extensions of K, and let $\mathcal{O}_L$ and $\mathcal{O}_{L'}$ be the respective integral closures of $\mathcal{O}_K$ in L and L'. Further, let M be a finite separable extension of L, and let $\mathcal{O}_M$ be the integral closure of $\mathcal{O}_L$ in M. Let $\mathfrak{p}$ be a nonzero prime of $\mathcal{O}_K$, let $\mathfrak{q}$ be a nonzero prime of $\mathcal{O}_L$ lying over $\mathfrak{p}$, and let $\mathfrak{l}$ be a nonzero prime of $\mathcal{O}_M$ lying over $\mathfrak{q}$.
   (a) Show that if $\mathfrak{p}$ is totally split in $\mathcal{O}_L$ and $\mathcal{O}_{L'}$, then it is also totally split in the composite extension LL'.
   (b) Show that if $\mathfrak{p}$ is totally ramified in $\mathcal{O}_M$ then $\mathfrak{p}$ is totally ramified in $\mathcal{O}_L$.
   (c) Show that if p is totally ramified in $\mathcal{O}_L$ and unramified in $\mathcal{O}_{L'}$, then $L \cap L' = K$.

(2) Let $\mathcal{O}_K$ be a Dedekind domain with field of fractions K. Let L be a finite separable extension of K. Show that, for every integral ideal $\mathfrak{a} \subset \mathcal{O}_L$, there exists a $\theta \in \mathcal{O}_L$ such that the conductor $\mathfrak{F}_\theta := \{x \in \mathcal{O}_L : x\mathcal{O}_L \subset \mathcal{O}_K[\theta]\}$ is prime to $\mathfrak{a}$ and such that $L = K(\theta)$.

(3) Let $f(x) \in \mathbb{Z}[x]$ be any nonconstant polynomial.
   (a) Prove that f has a root mod p for infinitely many primes p. [Hint: Prove this first under the assumption that $f(0) = 1$ by considering prime divisors of $f(n!)$. Reduce to this case by setting $g(x) = f(xf(0)))/f(0)$.]
   (b) Let K be a number field. Prove that there are infinitely many primes $\mathfrak{p}$ in $\mathcal{O}_K$ such that $f(\mathfrak{p}|p) = 1$, where $(p) = \mathfrak{p} \cap \mathbb{Z}$.
   (c) Prove that for $m \in \mathbb{Z}$ there are infinitely many primes $p \equiv 1 \bmod m$. [Please dont quote Dirichlets theorem: you are being asked to prove this special case!]
   (d) Prove that there are infinitely many primes $p \in \mathbb{Z}$ that split completely in $\mathcal{O}_K$. [Hint: apply (b) to the Galois closure of K.]
   (e) Suppose that $f(x)$ is irreducible. Prove that f splits into a product of linear factors over $\mathbb{F}_p$ for infinitely many primes p.

(4) Let L/K be an finite extension of number fields and let M be be a finite Galois extension of K containing L. Let $G := \mathrm{Gal}(M/K)$ and let $H := \mathrm{Gal}(M/L)$.
   (a) Let $\mathfrak{p} \subset \mathcal{O}_K$ be a nonzero prime, let $\mathfrak{q} \subset \mathcal{O}_L$ be a prime lying over $\mathfrak{p}$, and let $\mathfrak{l} \subset \mathcal{O}_M$ be a prime lying over $\mathfrak{q}$. Show that $e_{\mathfrak{q}/\mathfrak{p}} = [I_\mathfrak{l} : H \cap I_\mathfrak{l}]$ and that $f_{\mathfrak{q}/\mathfrak{p}} e_{\mathfrak{q}/\mathfrak{p}} = [D_\mathfrak{l} : H \cap D_\mathfrak{l}]$, where $D_\mathfrak{l}$ and $I_\mathfrak{l}$ denote the decomposition and inertia groups for $\mathfrak{l}/\mathfrak{p}$.
   (b) Let $\mathfrak{p} \subset \mathcal{O}_K$ be a nonzero prime and let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the set of primes of $\mathcal{O}_L$ lying over $\mathfrak{p}$. Prove that $f_{\mathfrak{q}_i/\mathfrak{p}} = f_{\mathfrak{q}_j/\mathfrak{p}}$ for all i, j if and only if $\#(H \cap D_\mathfrak{l}) = \#(H \cap D_{\sigma(\mathfrak{l})})$ for some $\mathfrak{l}$ lying above $\mathfrak{p}$ and all $\sigma \in \mathrm{Gal}(M/K)$. Similarly prove that $e_{\mathfrak{q}_i/\mathfrak{p}} = e_{\mathfrak{q}_j/\mathfrak{p}}$ for all i, j if and only if $\#(H \cap I_\mathfrak{l}) = \#(H \cap I_{\sigma(\mathfrak{l})})$ for some $\mathfrak{l}$ lying above $\mathfrak{p}$ and all $\sigma \in \mathrm{Gal}(M/K)$.

(c) Chebotarev's density theorem implies that for every $\sigma \in G$, there exist infinitely many primes $\mathfrak{l} \subset \mathcal{O}_M$ such that $\sigma = \mathrm{Frob}_{\mathfrak{l}}$. Use this together with parts (a) and (b) to prove that if $f_{\mathfrak{q}/\mathfrak{p}} = f_{\mathfrak{q}'/\mathfrak{p}}$ and $e_{\mathfrak{q}/\mathfrak{p}} = e_{\mathfrak{q}'/\mathfrak{p}}$ for all primes $\mathfrak{p}$ of $\mathcal{O}_K$ and all primes $\mathfrak{q}$ and $\mathfrak{q}'$ of $\mathcal{O}_L$ lying over $\mathfrak{p}$, then L/K is Galois.

(5) Compute the class group of $\mathbb{Q}(\zeta_{11})$. (You will need to use the Minkowski bound as well as information about cyclotomic fields. You should not just plug this in to some computer algebra software!)

(6) Prove that a subset L of $\mathbb{R}^n$ is a lattice if and only if it is a discrete subgroup of of $\mathbb{R}^n$.

University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195, USA

E-mail address: `bviray@math.washington.edu`

URL: `http://math.washington.edu/~bviray`