# MATH 581G: HOMEWORK ASSIGNMENT # 1

DUE MONDAY, OCTOBER 9

Problems 1.1, 1.4, 1.5, 1.6, 1.7 from Osserman's notes and the following.

(1) Let $K = \mathbb{Q}(\sqrt{7}, \sqrt{10})$. The goal of this problem is to prove that $\mathcal{O}_K \neq \mathbb{Z}[\alpha]$ for any $\alpha \in K$.

Let $\alpha \in K$, let $f(x)$ be the monic minimal polynomial of $\alpha$. For any $g(x) \in \mathbb{Z}[x]$, let $\overline{g}(x)$ be the image of $g$ in $\mathbb{Z}/3[x]$ under the natural map.

(a) Show that $g(\alpha)$ is divisible by 3 in $\mathbb{Z}[\alpha]$ if and only if $\overline{g}$ is divisible by $\overline{f}$ in $\mathbb{Z}/3[x]$.

(b) Consider the algebraic integers

$$\alpha_{ij} = (1 - (-1)^i\sqrt{7})(1 - (-1)^j\sqrt{10}).$$

Assume that $\mathcal{O}_K = \mathbb{Z}[\alpha]$. Show that the product of any two distinct $\alpha_{ij}$ is divisible by 3 in $\mathbb{Z}[\alpha]$, but that 3 does not divide any power of a single $\alpha_{ij}$.

(c) Assume that $\mathcal{O}_K = \mathbb{Z}[\alpha]$. Then $\alpha_{ij} = f_{ij}(\alpha)$ for some $f_{ij} \in \mathbb{Z}[x]$. Show that $\overline{f}$ divides the product of any two distinct $\overline{f}_{ij}$ but does not divide any power of a single $\overline{f}_{ij}$. Conclude that for all $(i,j),(k,l)$ with $(i,j) \neq (k,l)$, there is an irreducible factor that divides $\overline{f}$ and $\overline{f}_{ij}$, but not $\overline{f}_{kl}$.

(d) Conclude that $\overline{f}$ has at least four distinct irreducible factors over $\mathbb{Z}/3[x]$ and show that this results in a contradiction.

(2) Let $K = \mathbb{Q}(\alpha)$ be a number field of degree $n$. Let $f(x)$ be the monic minimal polynomial of $\alpha$ and let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ be the conjugates of $\alpha$. Prove that

$$\mathrm{disc}(1, \alpha, \ldots, \alpha^{n-1}) = (-1)^{\frac{n(n-1)}{2}} \mathrm{Norm}_{K/\mathbb{Q}}(f'(\alpha)).$$

Apply this to $\alpha = \zeta_p = e^{\frac{2\pi i}{p}}$ for a prime $p$ to show that

$$\mathrm{disc}(1, \zeta_p, \ldots, \zeta_p^{p-2}) = (-1)^{(p-1)(p-2)/2} p^{p-2}.$$

(3) Let $K$ be a number field of degree $n$ over $\mathbb{Q}$. Show that $\alpha_1, \ldots, \alpha_n$ is an integral basis if and only if $\mathrm{disc}(\alpha_1, \ldots, \alpha_n) = \mathrm{disc}(\mathcal{O}_K)$. Prove that if $\alpha_1, \ldots, \alpha_n \in \mathcal{O}_K$ and $\mathrm{disc}(\alpha_1, \ldots, \alpha_n)$ is squarefree, then $\alpha_1, \ldots, \alpha_n$ form an integral basis for $\mathcal{O}_K$.

(4) Let $f(x) = x^3 + ax + b$ with $a, b \in \mathbb{Z}$ and assume that $f(x)$ is irreducible over $\mathbb{Z}$. Let $\alpha \in \overline{\mathbb{Q}}$ be a root of $f(x)$.

(a) Show that $f'(\alpha) = -(2a\alpha + 3b)/\alpha$. Show that $2a\alpha + 3b$ is a root of

$$\left(\frac{x - 3b}{2a}\right)^3 + a\left(\frac{x - 3b}{2a}\right) + b.$$

Compute $\mathrm{Norm}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(2a\alpha + 3b)$.

(b) Show that $\mathrm{disc}(1, \alpha, \alpha^2) = -(4a^3 + 27b^2)$.

(c) Suppose that $\alpha^3 = \alpha + 1$ or that $\alpha^3 + \alpha = 1$. Prove that $\{1, \alpha, \alpha^2\}$ is an integral basis for $\mathcal{O}_{\mathbb{Q}(\alpha)}$.

(5) Let $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$. Then $K$ contains $\mathbb{Q}(\sqrt{mn})$.

(a) For $\alpha \in K$ show that $\alpha$ is integral if and only if $N_{K/\mathbb{Q}(\sqrt{m})}(\alpha)$ and $\mathrm{Tr}_{K/\mathbb{Q}(\sqrt{m})}$ are algebraic integers.

(b) Suppose that $m \equiv 3 \pmod 4$ and that $n \equiv 2 \pmod 4$. Show that every $\alpha \in \mathcal{O}_K$ has the form

$$\frac{a + b\sqrt{m} + c\sqrt{n} + d\sqrt{mn/\gcd(m,n)^2}}{2},$$

for some $a, b, c, d \in \mathbb{Z}$.

(c) Continuing from $(b)$, show that $a$ and $b$ must be even and that $c \equiv d \pmod 2$ by considering $\mathrm{Norm}_{K/\mathbb{Q}(\sqrt{m})}(\alpha)$. Conclude that

$$\left\{1, \sqrt{m}, \sqrt{n}, \frac{1}{2}\left(\sqrt{n} + \sqrt{mn/\gcd(m,n)^2}\right)\right\}$$

is an integral basis for $\mathcal{O}_K$.

(d) Show that $\mathrm{disc}(\mathcal{O}_K) = 64(mn)^2/\gcd(m,n)^2$.

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