As I mentioned in class, there are two main applications of the Fundamental Theorem of Finite Abelian Groups and its proof:

- Enumerating all abelian groups of order n for a fixed n, and
- Determining whether two abelian groups of order n are isomorphic.

I solve two types of these problems to demonstrate the technique. I suggest that you first attempt to solve these problems on your own, and then (and only then) look at the solution.

Enumerating all abelian groups of order n

Problem. Give a complete list of all abelian groups of order 144, no two of which are isomorphic.

Note that $144 = 2^4 \cdot 3^2$. By the Fundamental Theorem of Finite Abelian Groups, every abelian group of order 144 is isomorphic to the direct product of an abelian group of order $16 = 2^4$ and an abelian group of order $9 = 3^2$. Furthermore, abelian groups of order $16 = 2^4$, up to isomorphism, are in bijection with partitions of 4, and abelian groups of order $9 = 3^2$ are in bijection with partitions of 2. Thus, there are $5 \cdot 2 = 10$ abelian groups of order 144 and they are

$$\begin{array}{ll}
\mathbb{Z}_9 \times \mathbb{Z}_{16} & \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{16} \\
\mathbb{Z}_9 \times \mathbb{Z}_8 \times \mathbb{Z}_2 & \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_8 \times \mathbb{Z}_2 \\
\mathbb{Z}_9 \times \mathbb{Z}_4 \times \mathbb{Z}_4 & \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \\
\mathbb{Z}_9 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \\
\mathbb{Z}_9 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2
\end{array}$$

Enumerating all abelian groups of order n

Problem. Are $A := \mathbb{Z}_{30} \times \mathbb{Z}_{20} \times \mathbb{Z}_4$ and $B := \mathbb{Z}_{15} \times \mathbb{Z}_{10} \times \mathbb{Z}_{16}$ isomorphic?

We first prove the following lemmas (which are also in the book).

Lemma 0.1. *Let $m_1, m_2, \ldots, m_r$ be relatively prime positive integers. Then*

$$\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r} \cong \mathbb{Z}_{m_1 m_2 \cdots m_r}.$$

Proof. The element $(1, 1, \ldots, 1) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$ has order $\operatorname{lcm}(m_1, m_2, \ldots, m_r) = m_1 m_2 \cdots m_r$, which is the order of the group. Therefore, $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$ is cyclic and so is isomorphic to $\mathbb{Z}_{m_1 m_2 \cdots m_r}$. □

Lemma 0.2. *Let $G_1, G_1', G_2, G_2', \ldots, G_r, G_r'$ be groups such that for all i, $G_i \cong G_i'$. Then*

$$G_1 \times G_2 \times \cdots \times G_r \cong G_1' \times G_2' \times \cdots \times G_r'.$$

Proof. Let $\phi_i \colon G_i \to G_i'$ be an isomorphism. Then one can check that the map

$$\phi \colon G_1 \times G_2 \times \cdots \times G_r \to G_1' \times G_2' \times \cdots \times G_r', \quad \phi\left((g_1, g_2, \ldots, g_r)\right) = (\phi_1(g_1), \phi_2(g_2), \ldots, \phi_r(g_r))$$

is an isomorphism. $\square$

Now let us complete the problem. By Lemma 0.1, we have

$$\mathbb{Z}_{30} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5, \quad \text{and} \quad \mathbb{Z}_{20} \cong \mathbb{Z}_4 \times \mathbb{Z}_5.$$

Therefore, by Lemma 0.2 (and some rearranging), we have

$$A \cong (\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4) \times (\mathbb{Z}_3) \times (\mathbb{Z}_5 \times \mathbb{Z}_5). \tag{1}$$

Similarly,

$$\mathbb{Z}_{15} \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \quad \text{and} \quad \mathbb{Z}_{10} \cong \mathbb{Z}_2 \times \mathbb{Z}_5,$$

so

$$B \cong (\mathbb{Z}_2 \times \mathbb{Z}_{16}) \times (\mathbb{Z}_3) \times (\mathbb{Z}_5 \times \mathbb{Z}_5). \tag{2}$$

By the Fundamental Theorem of Finite Abelian Groups, there is a *unique* representation of any finite abelian group as a product of cyclic groups of prime power order. Since the cyclic groups of prime power order that appear in (1) and (2) are not the same, A is not isomorphic to B.