A FAMILY OF VARIETIES WITH EXACTLY ONE POINTLESS RATIONAL FIBER

BIANCA VIRAY

Abstract. We construct a concrete example of a 1-parameter family of smooth projective geometrically integral varieties over an open subscheme of $\mathbb{P}_\mathbb{Q}^1$ such that there is exactly one rational fiber with no rational points. This makes explicit a construction of Poonen.

1. Introduction

We construct a family of smooth projective geometrically integral surfaces over an open subscheme of $\mathbb{P}_\mathbb{Q}^1$ with the following curious arithmetic property: there is exactly one $\mathbb{Q}$-fiber with no rational points. Our proof makes explicit a non-effective construction of Poonen [6, Prop. 7.2], thus giving “an extreme example of geometry not controlling arithmetic” [6, p.2]. We believe that this is the first example of its kind.

Theorem 1.1. Define $P_0(x) := (x^2 - 2)(3 - x^2)$ and $P_\infty(x) := 2x^4 + 3x^2 - 1$. Let $\pi : X \to \mathbb{P}_\mathbb{Q}^1$ be the Châtelet surface bundle over $\mathbb{P}_\mathbb{Q}^1$ given by

$y^2 + z^2 = (6u^2 - v^2)^2 P_0(x) + (12v^2)^2 P_\infty(x),$

where $\pi$ is projection onto $(u : v)$. Then $\pi(X(\mathbb{Q})) = A^1_{\mathbb{Q}}(\mathbb{Q})$.

Note that the degenerate fibers of $\pi$ do not lie over $\mathbb{P}_\mathbb{Q}^1(\mathbb{Q})$ so the family of smooth projective geometrically integral surfaces mentioned above contains all $\mathbb{Q}$-fibers.

The non-effectivity in [6, Prop. 7.2] stems from the use of higher genus curves and Faltings’ theorem. (This is described in more detail in [6, §9]). We circumvent the use of higher genus curves by an appropriate choice of $P_\infty(x)$.

2. Background

This information can be found in [6, §3,5, and 6]. We review it here for the reader’s convenience.

Let $\mathcal{E}$ be a rank 3 vector sheaf on a $k$-variety $B$. A conic bundle $C$ over $B$ is the zero locus in $\mathbb{P}\mathcal{E}$ of a nowhere vanishing zero section $s \in \Gamma(\mathbb{P}\mathcal{E}, \text{Sym}^2(\mathcal{E}))$. A diagonal conic bundle is a conic bundle where $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ and $s = s_1 + s_2 + s_3, s_i \in \Gamma(\mathbb{P}\mathcal{E}, \mathcal{L}_i^{\otimes 2})$.

Now let $\alpha \in k^\times$, and let $P(x) \in k[x]$ be a separable polynomial of degree 3 or 4. Consider the diagonal conic bundle $X$ given by $B = \mathbb{P}^1, \mathcal{L}_1 = \mathcal{O}, \mathcal{L}_2 = \mathcal{O}, \mathcal{L}_3 = \mathcal{O}(2), s_1 = 1, s_2 = -\alpha, s_3 = -w^4 P(x/w)$. This smooth conic bundle contains the affine hypersurface $y^2 - \alpha z^2 = P(x) \subset \mathbb{A}^3$ as an open subscheme. We say that $X$ is the Châtelet surface given by

$y^2 - \alpha z^2 = P(x).$

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Note that since \( P(x) \) is not identically zero, \( X \) is an integral surface.

A Châtelet surface bundle over \( \mathbb{P}^1 \) is a flat proper morphism \( V \to \mathbb{P}^1 \) such that the generic fiber is a Châtelet surface. We can construct them in the following way. Let \( P, Q \in k[x, w] \) be linearly independent homogeneous polynomials of degree 4 and let \( \alpha \in k^x \). Let \( V \) be the diagonal conic bundle over \( \mathbb{P}^1_{(a:b)} \times \mathbb{P}^1_{(t:s)} \) given by \( L_1 = \mathcal{O}, L_2 = \mathcal{O}, L_3 = \mathcal{O}(1, 2), s_1 = 1, s_2 = -\alpha, s_3 = -(a^2P + b^2Q) \). By composing \( V \to \mathbb{P}^1 \times \mathbb{P}^1 \) with the projection onto the first factor, we realize \( V \) as a Châtelet surface bundle. We say that \( V \) is the Châtelet surface bundle given by

\[
y^2 - \alpha z^2 = a^2P(x) + b^2Q(x),
\]

where \( P(x) = P(x, 1) \) and \( Q(x) = Q(x, 1) \). We can also view \( a, b \) as relatively prime, homogeneous, degree \( d \) polynomials in \( u, v \) by pulling back by a suitable degree \( d \) map \( \phi: \mathbb{P}^1_{(u:v)} \to \mathbb{P}^1_{(a:b)} \).

3. Proof of Theorem 1.1

By [5], we know that the Châtelet surface

\[
y^2 + z^2 = (x^2 - 2)(3 - x^2)
\]

violates the Hasse principle, i.e. it has \( \mathbb{Q}_v \)-rational points for all completions \( v \), but no \( \mathbb{Q} \)-rational points. Thus, \( \pi(X(\mathbb{Q})) \subseteq \mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q}) \). Therefore, it remains to show that \( X_{(u,1)} \), the Châtelet surface defined by

\[
y^2 + z^2 = (6u^2 - 1)^2P_0(x) + 12^2P_{\infty}(x),
\]

has a rational point for all \( u \in \mathbb{Q} \).

If \( P_{(u:1)} := (6u^2 - 1)^2P_0(x) + 12^2P_{\infty}(x) \) is irreducible, then by [3], [4] we know that \( X_{(u,1)} \) satisfies the Hasse principle. Thus it suffices to show that \( P_{(u,1)} \) is irreducible and \( X_{(u,1)}(\mathbb{Q}_v) \neq \emptyset \) for all \( u \in \mathbb{Q} \) and all places \( v \) of \( \mathbb{Q} \).

3.1. Irreducibility. We prove that for any \( u \in \mathbb{Q} \), the polynomial \( P_{(u,1)}(x) \) is irreducible in \( \mathbb{Q}[x] \) by proving the slightly more general statement, that for all \( t \in \mathbb{Q} \)

\[
P_t(x) := (2x^4 + 3x^2 - 1) + t^2(x^2 - 2)(3 - x^2) = x^4(2 - t^2) + x^2(3 + 5t^2) + (-6t^2 - 1)
\]

is irreducible in \( \mathbb{Q}[x] \). We will use the fact that if \( a, b, c \in \mathbb{Q} \) are such that \( b^2 - 4ac \) and \( ac \) are not squares in \( \mathbb{Q} \) then \( p(x) := ax^4 + bx^2 + c \) is irreducible in \( \mathbb{Q}[x] \).

Let us first check that for all \( t \in \mathbb{Q}, (3 + 5t^2)^2 - 4(2 - t^2)(-6t^2 - 1) \) is not a square in \( \mathbb{Q} \). This is equivalent to proving that the affine curve \( C: w^2 = t^4 + 74t^2 + 17 \) has no rational points. The smooth projective model, \( \overline{C} : w^2 = t^4 + 74t^2s^2 + 17s^4 \), has 2 rational points at infinity. Therefore \( \overline{C} \) is isomorphic to its Jacobian. A computation in Magma shows that \( \text{Jac}(\overline{C})(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \) [1]. Therefore, the points at infinity are the only 2 rational points of \( \overline{C} \) and thus \( C \) has no rational points.

Now we will show that \( (-6t^2 - 1)(2 - t^2) \) is not a square in \( \mathbb{Q} \) for any \( t \in \mathbb{Q} \). As above, this is equivalent to determining whether \( C' : w^2 = (-6t^2 - 1)(2 - t^2) \) has a rational point. Since \( 6 \) is not a square in \( \mathbb{Q} \), this is equivalent to determining whether the smooth projective model, \( \overline{C'} \), has a rational point. The curve \( \overline{C'} \) is a genus 1 curve so it is either isomorphic to its Jacobian or has no rational points. A computation in Magma shows that \( \text{Jac}(\overline{C'})(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \) [1]. Thus \( \#C'(\mathbb{Q}) = 0 \) or 2. If \( (t, w) \) is a rational point of \( C' \), then \( (\pm t, \pm w) \) is also
a rational point. Therefore, \( \#C(\mathbb{Q}) = 2 \) if and only if there is a point with \( t = 0 \) or \( w = 0 \) and one can easily check that this is not the case.

### 3.2. Local Solvability.

**Lemma 3.1.** For any point \( (u : v) \in \mathbb{P}^1 \), the Châtelet surface \( X_{(u,v)} \) has \( \mathbb{R} \)-points and \( \mathbb{Q}_p \)-points for every prime \( p \).

**Proof.** Let \( a = 6u^2 - v^2 \) and let \( b = 12v^2 \). We will refer to \( a^2P_0(x) + b^2P_\infty(x) \) both as \( P_{(a,b)} \) and \( P_{(a,v)} \).

**R-points:** It suffices to show that given \( (u : v) \) there exists an \( x \) such that
\[
P_{(a,b)} = x^4(2b^2 - a^2) + x^2(3b^2 + 5a^2) + (-6a^2 - b^2)
\]
is positive. If \( 2b^2 - a^2 \) is positive, then any \( x \) sufficiently large will work. So assume \( 2b^2 - a^2 \) is negative. Then \( \alpha = \frac{-3b^2 + 5a^2}{2(2b^2 - a^2)} \) is positive. We claim \( P_{(a,b)}(\sqrt{\alpha}) \) is positive.

\[
P_{(a,b)}(\sqrt{\alpha}) = \alpha^2(2b^2 - a^2) + \alpha(3b^2 + 5a^2) + (-6a^2 - b^2) = \frac{(3b^2 + 5a^2)^2}{4(2b^2 - a^2)} + \frac{-3b^2 + 5a^2}{2(2b^2 - a^2)} + (-6a^2 - b^2)
\]
\[
= \frac{1}{4(2b^2 - a^2)}(4(2b^2 - a^2)(-6a^2 - b^2) - (3b^2 + 5a^2)^2)
\]
\[
= \frac{1}{4(2b^2 - a^2)}(-17b^4 - 74a^2b^2 - a^4)
\]

Since \( 2b^2 - a^2 \) is negative by assumption and \(-17b^4 - 74a^2b^2 - a^4 \) is always negative, we have our result.

**\( \mathbb{Q}_p \)-points:**

\( p \geq 5 \): Without loss of generality, let \( a \) and \( b \) be relatively prime integers. Let \( \overline{X}_{(a,b)} \) denote the reduction of \( X_{(a,b)} \) modulo \( p \). We claim that there exists a smooth \( \mathbb{F}_p \)-point of \( \overline{X}_{(a,b)} \) that, by Hensel’s lemma, we can lift to a \( \mathbb{Q}_p \)-point of \( X_{(a,b)} \).

Since \( P_{(a,b)} \) has degree at most 4 and is not identically zero modulo \( p \), there is some \( x \in \mathbb{F}_p \) such that \( P_{(a,b)}(x) \) is nonzero. Now let \( y, z \) run over all values in \( \mathbb{F}_p \). Then the polynomials \( y^2, P_{(a,b)}(x) - z^2 \) each take \( (p + 1)/2 \) distinct values. By the pigeonhole principle, \( y^2 \) and \( P_{(a,b)}(x) - z^2 \) must agree for at least one pair \( (y, z) \in \mathbb{F}_p^2 \) and one can check that this pair is not \((0, 0)\). Thus, this tuple \((x, y, z)\) gives a smooth \( \mathbb{F}_p \)-point of \( \overline{X}_{(a,b)} \). (The proof above that the quadratic form \( y^2 + z^2 \) represents any element in \( \mathbb{F}_p \) is not new. For example, it can be found in [2, Prop 5.2.1].)

\( p = 3 \): From the equations for \( a \) and \( b \), one can check that for any \( (u : v) \in \mathbb{P}^1 \), \( v_3(b/a) \) is positive. Since \( \mathbb{Q}_3(\sqrt{-1})/\mathbb{Q}_3 \) is an unramified extension, it suffices to show that given \( a, b \) as above, there exists an \( x \) such that \( P_{(a,b)}(x) \) has even valuation. Since \( v_3(b/a) \) is positive, \( v_3(2b^2 - a^2) = 2v_3(a) \). Therefore, if \( x = 3^{-n} \), for \( n \) sufficiently large, the valuation of \( P_{(a,b)}(x) \) is \(-4n + 2v_3(a)\) which is even.

\( p = 2 \): From the equations for \( a \) and \( b \), one can check that for any \( (u : v) \in \mathbb{P}^1 \), \( v_2(b/a) \) is at least 2. Let \( x = 0 \) and \( y = a \). Then we need to find a solution to \( z^2 = a^2(-7 + (b/a)^2) \). Since \( v_2(b/a) > 1, -7 + (b/a)^2 \equiv 1^2 \mod 8 \). By Hensel’s lemma, we can lift this to a solution in \( \mathbb{Q}_2 \).

\[ \square \]
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References


Bianca Viray, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
E-mail address: bviray@math.berkeley.edu
URL: http://math.berkeley.edu/~bviray