A FAMILY OF VARIETIES WITH EXACTLY ONE POINTLESS RATIONAL FIBER

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Abstract. We construct a concrete example of a 1-parameter family of smooth projective geometrically integral varieties over an open subscheme of $\mathbb{P}^1_{\mathbb{Q}}$ such that there is exactly one rational fiber with no rational points. This makes explicit a construction of Poonen.

1. Introduction

We construct a family of smooth projective geometrically integral surfaces over an open subscheme of $\mathbb{P}^1_{\mathbb{Q}}$ with the following curious arithmetic property: there is exactly one $\mathbb{Q}$-fiber with no rational points. Our proof makes explicit a non-effective construction of Poonen [6, Prop. 7.2], thus giving "an extreme example of geometry *not* controlling arithmetic" [6, p.2]. We believe that this is the first example of its kind.

Theorem 1.1. *Define* $P_0(x) := (x^2-2)(3-x^2)$ *and* $P_\infty(x) := 2x^4+3x^2-1$. *Let* $\pi\colon X \to \mathbb{P}^1_{\mathbb{Q}}$ *be the Châtelet surface bundle over* $\mathbb{P}^1_{\mathbb{Q}}$ *given by*

$$y^2 + z^2 = \left(6u^2 - v^2\right)^2 P_0(x) + \left(12v^2\right)^2 P_\infty(x),$$

where π *is projection onto* $(u : v)$. *Then* $\pi(X(\mathbb{Q})) = \mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q})$.

Note that the degenerate fibers of π do not lie over $\mathbb{P}^1(\mathbb{Q})$ so the family of smooth projective geometrically integral surfaces mentioned above contains all $\mathbb{Q}$-fibers.

The non-effectivity in [6, Prop. 7.2] stems from the use of higher genus curves and Faltings' theorem. (This is described in more detail in [6, §9]). We circumvent the use of higher genus curves by an appropriate choice of $P_\infty(x)$.

2. Background

This information can be found in [6, §3,5, and 6]. We review it here for the reader's convenience.

Let $\mathcal{E}$ be a rank 3 vector sheaf on a k-variety B. A conic bundle C over B is the zero locus in $\mathbb{P}\mathcal{E}$ of a nowhere vanishing zero section $s \in \Gamma(\mathbb{P}\mathcal{E}, \mathrm{Sym}^2(\mathcal{E}))$. A diagonal conic bundle is a conic bundle where $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ and $s = s_1 + s_2 + s_3, s_i \in \Gamma(\mathbb{P}\mathcal{E}, \mathcal{L}_i^{\otimes 2})$.

Now let $\alpha \in k^\times$, and let $P(x) \in k[x]$ be a separable polynomial of degree 3 or 4. Consider the diagonal conic bundle X given by $B = \mathbb{P}^1, \mathcal{L}_1 = \mathcal{O}, \mathcal{L}_2 = \mathcal{O}, \mathcal{L}_3 = \mathcal{O}(2), s_1 = 1, s_2 = -\alpha, s_3 = -w^4P(x/w)$. This smooth conic bundle contains the affine hypersurface $y^2 - \alpha z^2 = P(x) \subset \mathbb{A}^3$ as an open subscheme. We say that X is the Châtelet surface given by

$$y^2 - \alpha z^2 = P(x).$$

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Note that since $P(x)$ is not identically zero, X is an integral surface.

A Châtelet surface bundle over $\mathbb{P}^1$ is a flat proper morphism $V \to \mathbb{P}^1$ such that the generic fiber is a Châtelet surface. We can construct them in the following way. Let $P, Q \in k[x, w]$ be linearly independent homogeneous polynomials of degree 4 and let $\alpha \in k^\times$. Let V be the diagonal conic bundle over $\mathbb{P}^1_{(a:b)} \times \mathbb{P}^1_{(w:x)}$ given by $\mathcal{L}_1 = \mathcal{O}, \mathcal{L}_2 = \mathcal{O}, \mathcal{L}_3 = \mathcal{O}(1,2), s_1 = 1, s_2 = -\alpha, s_3 = -(a^2P + b^2Q)$. By composing $V \to \mathbb{P}^1 \times \mathbb{P}^1$ with the projection onto the first factor, we realize V as a Châtelet surface bundle. We say that V is the Châtelet surface bundle given by

$$y^2 - \alpha z^2 = a^2P(x) + b^2Q(x),$$

where $P(x) = P(x, 1)$ and $Q(x) = Q(x, 1)$. We can also view a, b as relatively prime, homogeneous, degree d polynomials in u, v by pulling back by a suitable degree d map $\phi\colon \mathbb{P}^1_{(u:v)} \to \mathbb{P}^1_{(a:b)}$.

3. Proof of Theorem 1.1

By [5], we know that the Châtelet surface

$$y^2 + z^2 = (x^2 - 2)(3 - x^2)$$

violates the Hasse principle, i.e. it has $\mathbb{Q}_v$-rational points for all completions v, but no $\mathbb{Q}$-rational points. Thus, $\pi(X(\mathbb{Q})) \subseteq \mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q})$. Therefore, it remains to show that $X_{(u:1)}$, the Châtelet surface defined by

$$y^2 + z^2 = (6u^2 - 1)^2 P_0(x) + 12^2 P_\infty(x),$$

has a rational point for all $u \in \mathbb{Q}$.

If $P_{(u:1)} := (6u^2 - 1)^2 P_0(x) + 12^2 P_\infty(x)$ is irreducible, then by [3], [4] we know that $X_{(u:1)}$ satisfies the Hasse principle. Thus it suffices to show that $P_{(u:1)}$ is irreducible and $X_{(u:1)}(\mathbb{Q}_v) \neq \emptyset$ for all $u \in \mathbb{Q}$ and all places v of $\mathbb{Q}$.

3.1. Irreducibility.

We prove that for any $u \in \mathbb{Q}$, the polynomial $P_{(u:1)}(x)$ is irreducible in $\mathbb{Q}[x]$ by proving the slightly more general statement, that for all $t \in \mathbb{Q}$

$$P_t(x) := (2x^4 + 3x^2 - 1) + t^2(x^2 - 2)(3 - x^2) = x^4(2 - t^2) + x^2(3 + 5t^2) + (-6t^2 - 1)$$

is irreducible in $\mathbb{Q}[x]$. We will use the fact that if $a, b, c \in \mathbb{Q}$ are such that $b^2 - 4ac$ and ac are not squares in $\mathbb{Q}$ then $p(x) := ax^4 + bx^2 + c$ is irreducible in $\mathbb{Q}[x]$.

Let us first check that for all $t \in \mathbb{Q}$, $(3 + 5t^2)^2 - 4\,(2 - t^2)\,(-6t^2 - 1)$ is not a square in $\mathbb{Q}$. This is equivalent to proving that the affine curve $C\colon w^2 = t^4 + 74t^2 + 17$ has no rational points. The smooth projective model, $\overline{C} : w^2 = t^4 + 74t^2s^2 + 17s^4$ in weighted projective space $\mathbb{P}(1, 1, 2)$, has 2 rational points at infinity. Therefore $\overline{C}$ is isomorphic to its Jacobian. A computation in `Magma` shows that $\mathrm{Jac}(C)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ [1]. Therefore, the points at infinity are the only 2 rational points of $\overline{C}$ and thus C has no rational points.

Now we will show that $(-6t^2 - 1)\,(2 - t^2)$ is not a square in $\mathbb{Q}$ for any $t \in \mathbb{Q}$. As above, this is equivalent to determining whether $C'\colon w^2 = (-6t^2 - 1)(2 - t^2)$ has a rational point. Since 6 is not a square in $\mathbb{Q}$, this is equivalent to determining whether the smooth projective model, $\overline{C'}$, has a rational point. The curve $\overline{C'}$ is a genus 1 curve so it is either isomorphic to its Jacobian or has no rational points. A computation in `Magma` shows that $\mathrm{Jac}\,(C')\,(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ [1]. Thus $\#C'\,(\mathbb{Q}) = 0$ or 2. If (t, w) is a rational point of C', then $(\pm t, \pm w)$ is also a rational point. Therefore, $\#C(\mathbb{Q}) = 2$ if and only if there is a point with $t = 0$ or $w = 0$ and one can easily check that this is not the case.

3.2. **Local Solvability.**

Lemma 3.1. *For any point $(u : v) \in \mathbb{P}^1_{\mathbb{Q}}$, the Châtelet surface $X_{(u:v)}$ has $\mathbb{R}$-points and $\mathbb{Q}_p$-points for every prime p.*

Proof. Let $a = 6u^2 - v^2$ and let $b = 12v^2$. We will refer to $a^2P_0(x) + b^2P_\infty(x)$ both as $P_{(a:b)}$ and $P_{(u:v)}$.

$\mathbb{R}$-points: It suffices to show that given $(u : v)$ there exists an x such that

$$P_{(a:b)} = x^4(2b^2 - a^2) + x^2(3b^2 + 5a^2) + (-6a^2 - b^2)$$

is positive. If $2b^2 - a^2$ is positive, then any x sufficiently large will work. So assume $2b^2 - a^2$ is negative. Then $\alpha = \frac{-(3b^2+5a^2)}{2(2b^2-a^2)}$ is positive. We claim $P_{(a:b)}(\sqrt{\alpha})$ is positive.

$$\begin{aligned} P_{(a:b)}(\sqrt{\alpha}) &= \alpha^2(2b^2 - a^2) + \alpha(3b^2 + 5a^2) + (-6a^2 - b^2) \\ &= \frac{(3b^2 + 5a^2)^2}{4(2b^2 - a^2)} + \frac{-(3b^2 + 5a^2)^2}{2(2b^2 - a^2)} + (-6a^2 - b^2) \\ &= \frac{1}{4(2b^2 - a^2)}\left(4(2b^2 - a^2)(-6a^2 - b^2) - (3b^2 + 5a^2)^2\right) \\ &= \frac{1}{4(2b^2 - a^2)}\left(-17b^4 - 74a^2b^2 - a^4\right) \end{aligned}$$

Since $2b^2 - a^2$ is negative by assumption and $-17b^4 - 74a^2b^2 - a^4$ is always negative, we have our result.

$\mathbb{Q}_p$-points:

- **$p \geq 5$:** Without loss of generality, let a and b be relatively prime integers. Let $\overline{X}_{(a:b)}$ denote the reduction of $X_{(a:b)}$ modulo p. We claim that there exists a smooth $\mathbb{F}_p$-point of $\overline{X}_{(a:b)}$ that, by Hensel's lemma, we can lift to a $\mathbb{Q}_p$-point of $X_{(a:b)}$.

  Since $P_{(a:b)}$ has degree at most 4 and is not identically zero modulo p, there is some $x \in \mathbb{F}_p$ such that $P_{(a:b)}(x)$ is nonzero. Now let y, z run over all values in $\mathbb{F}_p$. Then the polynomials $y^2, P_{(a:b)}(x) - z^2$ each take $(p+1)/2$ distinct values. By the pigeonhole principle, y^2 and $P_{(a:b)}(x) - z^2$ must agree for at least one pair $(y, z) \in \mathbb{F}_p^2$ and one can check that this pair is not $(0, 0)$. Thus, this tuple (x, y, z) gives a smooth $\mathbb{F}_p$-point of $\overline{X}_{(a:b)}$. (The proof above that the quadratic form $y^2 + z^2$ represents any element in F_p is not new. For example, it can be found in [2, Prop 5.2.1].)
- **$p = 3$:** From the equations for a and b, one can check that for any $(u : v) \in \mathbb{P}^1_{\mathbb{Q}}$, $v_3(b/a)$ is positive. Since $\mathbb{Q}_3(\sqrt{-1})/\mathbb{Q}_3$ is an unramified extension, it suffices to show that given a, b as above, there exists an x such that $P_{(a:b)}(x)$ has even valuation. Since $v_3(b/a)$ is positive, $v_3(2b^2 - a^2) = 2v_3(a)$. Therefore, if $x = 3^{-n}$, for n sufficiently large, the valuation of $P_{(a:b)}(x)$ is $-4n + 2v_3(a)$ which is even.
- **$p = 2$:** From the equations for a and b, one can check that for any $(u : v) \in \mathbb{P}^1_{\mathbb{Q}}$, $v_2(b/a)$ is at least 2. Let $x = 0$ and $y = a$. Then we need to find a solution to $z^2 = a^2(-7 + (b/a)^2)$. Since $v_2(b/a) > 1$, $-7 + (b/a)^2 \equiv 1^2 \mod 8$. By Hensel's lemma, we can lift this to a solution in $\mathbb{Q}_2$.

$\square$

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