FAILURE OF THE HASSE PRINCIPLE FOR ENRIQUES SURFACES

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Abstract. We construct an Enriques surface X over $\mathbb{Q}$ with empty étale-Brauer set (and hence no rational points) for which there is no algebraic Brauer-Manin obstruction to the Hasse principle. In addition, if there is a transcendental obstruction on X, then we obtain a K3 surface that has a transcendental obstruction to the Hasse principle.

1. Introduction

Let X be a smooth, projective, geometrically integral scheme over a number field k. We say that X satisfies the Hasse principle if the set $X(k)$ of k-rational points is nonempty whenever the set of adelic points $X(\mathbb{A}_k)$ is also nonempty. Manin and Skorobogatov have defined intermediate "obstruction sets" that fit between $X(k)$ and $X(\mathbb{A}_k)$ (cf. §2 or [Man71, Sko99]):

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{et},\mathrm{Br}} \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k)^{\mathrm{Br}_1} \subseteq X(\mathbb{A}_k). \tag{1.1}$$

Lind and Reichardt, Harari, Skorobogatov, and Poonen constructed the first schemes that show, respectively, that each above containment (from right to left) can be strict [Lin40, Rei42, Har96, Sko99, Poo10].

A wide open area of research considers the finer question: to what extent do the sets in (1.1) give distinct obstructions to the Hasse principle *after* fixing some numerical invariants, like dimension, of X? For curves, Scharaschkin and Skorobogatov independently asked if $X(\mathbb{A}_k)^{\mathrm{Br}_1} \neq \emptyset$ implies that $X(k) \neq \emptyset$, i.e., if the algebraic Brauer-Manin obstruction explains all counterexamples to the Hasse principle [Sch99, Sko01]. (This question has since been upgraded to a conjecture [Fly04, Poo06, Sto07].) In the case of geometrically rational surfaces, Colliot-Thélène and Sansuc conjectured that the same implication holds [CTS80]. In contrast, for most other Enriques-Kodaira classes of surfaces we expect that this is no longer the case.

However, there are strikingly few examples of surfaces that corroborate this expectation. In a pioneering paper, Skorobogatov constructs the first surface for which the failure of the Hasse principle is not explained by an algebraic Brauer-Manin obstruction (or, for that matter, a transcendental Brauer-Manin obstruction) [Sko99]; the other known example, due to Basile and Skorobogatov, is of a similar nature [BS03]. In both cases, the surfaces considered are bi-elliptic and the failure is caused by an étale-Brauer obstruction, i.e. $X(\mathbb{A}_k)^{\mathrm{et},\mathrm{Br}} = \emptyset$.

We show that Enriques surfaces give rise to a similar insufficiency phenomenon. More precisely, our main result is as follows.

Theorem 1.1. *There exists an Enriques surface $X/\mathbb{Q}$ such that*

$$X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{et},\mathrm{Br}} = \emptyset \quad \text{and} \quad X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} \neq \emptyset.$$

Moreover, if $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$, then $Y(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}\, Y \setminus \mathrm{Br}_1 Y} = \emptyset$, where Y is a K3 double cover of X.

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Note the rather curious dichotomy: we obtain either a K3 surface with a transcendental Brauer-Manin obstruction to the Hasse principle, or an Enriques surface whose failure of the Hasse principle is unaccounted for by a Brauer-Manin obstruction.

It is important to remark that Cunnane already showed that Enriques surfaces need not satisfy the Hasse principle [Cun07]; however, his counterexamples are explained by an algebraic Brauer-Manin obstruction. On the other hand, Harari and Skorobogatov, and later Cunnane, showed that the Brauer-Manin obstruction is insufficient to explain all failures of weak approximation on Enriques surfaces [HS05, Cun07], thereby opening up the analogous question for the Hasse principle. Theorem 1.1 represents a key step towards a complete answer to this question.

1.1. **Outline of proof.** The proof of Theorem 1.1 is constructive. Let $\mathbf{a} := (a, b, c) \in \mathbb{Z}^3$, and consider the intersection $Y_{\mathbf{a}}$ of the three quadrics

$$\begin{aligned} xy + 5z^2 &= s^2 \\ (x+y)(x+2y) &= s^2 - 5t^2 \\ ax^2 + by^2 + cz^2 &= u^2. \end{aligned}$$

in $\mathbb{P}^5 = \operatorname{Proj} \mathbb{Q}[s, t, u, x, y, w]$. Suppose that

$$abc(5a + 5b + c)(20a + 5b + 2c)(4a^2 + b^2)(c^2 - 100ab)(c^2 + 5bc + 10ac + 25ab) \neq 0.$$

Then $Y_{\mathbf{a}}$ is smooth and thus defines a K3 surface. The involution

$$\sigma : \mathbb{P}^5 \to \mathbb{P}^5, \qquad (s : t : u : x : y : z) \mapsto (-s : -t : -u : x : y : z)$$

has no fixed points when restricted to $Y_{\mathbf{a}}$, so $X_{\mathbf{a}} := Y_{\mathbf{a}}/\sigma$ is an Enriques surface.

Theorem 1.2. *Let $\mathbf{a} = (a, b, c) \in \mathbb{Z}^3_{>0}$ satisfy the following conditions:*

1. *for all prime numbers $p \mid (5a + 5b + c)$, 5 is not a square modulo p,*
2. *for all prime numbers $p \mid (20a + 5b + 2c)$, 10 is not a square modulo p,*
3. *the quadratic form $ax^2 + by^2 + cz^2 + u^2$ is anisotropic over $\mathbb{Q}_3$,*
4. *the integer $-bc$ is not a square modulo 5,*
5. *the triplet (a, b, c) is congruent to $(5, 6, 6)$ modulo 7,*
6. *the triplet (a, b, c) is congruent to $(1, 1, 2)$ modulo 11,*
7. *$Y_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$, and*
8. *the triplet (a, b, c) is Galois general, meaning that a certain number field defined in terms of a, b, c is as large as possible. A precise definition is given in §4.1.2.*

Then

$$X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{et,Br}} = \emptyset \quad \text{and} \quad X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} \neq \emptyset.$$

Moreover, if $X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$, then $Y_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br} \setminus \mathrm{Br}_1} = \emptyset$.

Theorem 1.1 follows almost at once from Theorem 1.2: we show that the triplet $\mathbf{a} = (12, 111, 13)$ satisfies conditions (1)–(8).

Remarks 1.3.

(i) In lieu of a paper outline, let us explain the role that the assumptions of Theorem 1.2 play in our constructions. In §3, we show that conditions (1)–(4) imply the étale-Brauer set of $X_{\mathbf{a}}$ is empty. In §4, we use conditions (5), (6), and (8) to describe explicit generators for the Picard groups of $X_{\mathbf{a}}$ and $Y_{\mathbf{a}}$, as well as to compute the low degree Galois cohomology of these groups. In §5, we use the results from §4 together with conditions (5)–(8) to determine both the Brauer set and algebraic Brauer set of $X_{\mathbf{a}}$. Finally, in §6, we prove Theorems 1.2 and 1.1.

(ii) Our construction relies heavily on an example due to Birch and Swinnerton-Dyer of a del Pezzo surface of degree 4 that violates the Hasse principle [BSD75]. While we expect the construction will work with other such surfaces, there are a few subtleties which are not readily apparent in the argument. We elaborate on this point in §6.2.

1.2. **Notation.** Throughout k denotes a perfect field, $\overline{k}$ is a fixed algebraic closure and G_k denotes the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. For any k-scheme X, we write $\overline{X}$ for the base change $X \times_k \overline{k}$ and $\mathbf{k}(X)$ for the function field of X. For a smooth, projective, geometrically integral variety X we identify $\operatorname{Pic} X$ with the Weil class group; in particular, we use additive notation for the group law on $\operatorname{Pic} X$. In addition, we write K_X for the class of the canonical sheaf in the group $\operatorname{Pic} X$. Finally, we denote by $\rho(X)$ the geometric Picard number of X, i.e., the rank of the Néron-Severi group of $\overline{X}$.

For a homogeneous ideal I in a graded ring R, we write $V(I)$ for $\operatorname{Proj} R/I$; if explicit generators of $I = \langle f_i : i \in S\rangle$ are given, then we write $V(f_i : i \in S)$ instead of $V(I)$.

Henceforth, "condition(s)" refers to items (1) – (8) in Theorem 1.2.

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2. Background

2.1. **K3 surfaces and Enriques surfaces.** Assume that the characteristic of k is not 2. A K3 surface is a smooth projective k-surface X of Kodaira dimension 0 with trivial canonical divisor and $h^1(X, \mathcal{O}_X) = 0$. The geometric Picard group of a K3 surface is a free abelian group of rank at most 22 (in characteristic 0, the rank is at most 20). In addition, the intersection lattice of a K3 surface can be embedded primitively in $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, the unique even unimodular lattice of signature $(3, 19)$.

An Enriques surface is a smooth projective k-surface X of Kodaira dimension 0 with numerically trivial canonical divisor and second Betti number $b_2 = 10$. These assumptions imply that $K_X \neq 0$ and $2K_X = 0$. Enriques surfaces are also characterized in terms of K3 surfaces as follows [CD89, Proposition 1.3.1] [Bea96, Proposition VIII.17].

Theorem 2.1. *Let X be an Enriques surface and let $f\colon Y \to X$ be any étale double cover associated to $K_X \in (\operatorname{Pic} X)[2]$. Then Y is a K3 surface. Conversely, the quotient of a K3 surface by a fixed-point free involution is an Enriques surface.*

2.1.1. *Geometric birational models of Enriques surfaces.* Assume that k is algebraically closed. A generic Enriques surface can be constructed as follows. Consider a K3 surface Y of degree 8 in $\mathbb{P}^5_k = \operatorname{Proj} k[s,t,u,x,y,z]$ given by

$$V\big(Q_i(s,t,u) - \widetilde{Q}_i(x,y,z) : \quad i = 1,2,3\big), \tag{2.1}$$

where $Q_i \in k[s,t,u]$ and $\widetilde{Q}_i \in k[x,y,z]$ are quadratic polynomials for $i = 1,2,3$. For generic Q_i and $\widetilde{Q}_i$, the involution σ of §1.1 has no fixed points when restricted to Y, so $X := Y/\sigma$ is an Enriques surface. Cossec and Verra show that a stronger statement holds:

Let $f\colon Y \to X$ be the double cover of an Enriques surface X by a $K3$ surface Y. Then there exists a birational map between Y and a surface of the form (2.1) that identifies σ with the unique fixed-point free involution ι such that $f \circ \iota = f$ [Cos83, Ver83].

2.1.2. *The Picard group.* For any Enriques surface X, we have the following exact sequence of Galois modules [CD89, Theorem 1.2.1 and Proposition 1.2.1]

$$0 \to \langle K_X \rangle \to \operatorname{Pic} \overline{X} \to \operatorname{Num} \overline{X} \to 0. \tag{2.2}$$

Additionally, $\operatorname{Num} \overline{X} \cong U \oplus E_8(-1)$, the unique even unimodular lattice of rank 10 and signature $(1, 9)$ [CD89, Theorem 2.5.1]. We also have an exact sequence relating the geometric Picard group of X and that of any K3 double cover Y of X [HS05, Equation (3.15)]

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \operatorname{Pic} \overline{X} \to \left(\operatorname{Pic} \overline{Y}\right)^{\sigma} \to 0. \tag{2.3}$$

Combining (2.2) and (2.3) we obtain an isomorphism of Galois modules $\operatorname{Num} \overline{X} \cong \left(\operatorname{Pic} \overline{Y}\right)^{\sigma}$.

2.2. **The Brauer group and the Brauer-Manin obstruction.** For the remainder of §2, we restrict to the case where k is a number field. Let $\operatorname{Br} X := \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)$ be the Brauer group of a variety X. An element $\mathcal{A} \in \operatorname{Br} X$ is algebraic if it belongs to the subgroup $\operatorname{Br}_1 X := \ker\left(\operatorname{Br} X \to \operatorname{Br} \overline{X}\right)$; otherwise $\mathcal{A}$ is called transcendental.

Functoriality of the Brauer group yields an evaluation pairing

$$\langle \cdot , \cdot \rangle \colon X(\mathbb{A}_k) \times \operatorname{Br} X \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

For any set $S \subseteq \operatorname{Br} X$ (S need not be a subgroup), this pairing is used to define the set

$$X(\mathbb{A}_k)^S = \left\{ (P_v)_v \in X(\mathbb{A}_k) : \langle (P_v), \mathcal{A} \rangle = 0 \text{ for all } \mathcal{A} \in S \right\}.$$

Class field theory guarantees that $X(k) \subseteq X(\mathbb{A}_k)^S$, for any S. When $S = \operatorname{Br} X$ and $\operatorname{Br}_1 X$, respectively, we obtain the Brauer set of X and the algebraic Brauer set of X; we denote these sets by $X(\mathbb{A}_k)^{\operatorname{Br}}$ and $X(\mathbb{A}_k)^{\operatorname{Br}_1}$. In summary, we have

$$X(k) \subseteq X(\mathbb{A}_k)^{\operatorname{Br}} \subseteq X(\mathbb{A}_k)^{\operatorname{Br}_1} \subseteq X(\mathbb{A}_k).$$

We say there is a Brauer-Manin obstruction to the Hasse principle if $X(\mathbb{A}_k)^{\operatorname{Br}} = \emptyset$ and $X(\mathbb{A}_k) \neq \emptyset$. The obstruction is algebraic if in addition $X(\mathbb{A}_k)^{\operatorname{Br}_1} = \emptyset$. See [Sko01, §5.2] for more details.

2.3. **Torsors under finite étale groups and the étale-Brauer obstruction.** Let G be an fppf group scheme over a scheme X. Recall that a (right) G-torsor over X is an fppf X-scheme Y equipped with a right G-action such that the morphism $Y \times_X G \to Y \times_X Y$ given by $(y, g) \mapsto (y, yg)$ is an isomorphism. A detailed account of torsors is given in [Sko01, Part I]. A torsor $f\colon Y \to X$ under a finite étale k-group scheme G determines a partition

$$X(k) = \bigcup_{\tau \in \mathrm{H}^1(k,G)} f^{\tau}\left(Y^{\tau}(k)\right),$$

where Y^{τ} is the twisted torsor associated to τ (see [Sko01, Lemma 2.2.3]). Running over all possible G-torsors of this form, we assemble the étale-Brauer set

$$X(\mathbb{A}_k)^{\mathrm{et},\operatorname{Br}} := \bigcap_{\substack{f\colon Y \to X \\ \text{torsor under} \\ \text{finite étale } G}} \left(\bigcup_{\tau \in \mathrm{H}^1(k,G)} f^{\tau}\left(Y^{\tau}(\mathbb{A}_k)^{\operatorname{Br}}\right) \right).$$

By construction, we have $X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{et,Br}}$; we say there is an étale-Brauer obstruction to the Hasse principle if $X(\mathbb{A}_k) \neq \emptyset$ and $X(\mathbb{A}_k)^{\mathrm{et,Br}} = \emptyset$. Note that $X(\mathbb{A}_k)^{\mathrm{et,Br}} \subseteq X(\mathbb{A}_k)^{\mathrm{Br}}$, so the étale-Brauer obstruction is at least as strong as the Brauer-Manin obstruction.

2.3.1. *K3 double covers as torsors.* Any K3 double cover $f\colon Y \to X$ of an Enriques surface X is a $\mathbb{Z}/2\mathbb{Z}$-torsor over X. By Kummer theory, we have $\mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z}) = k^\times/(k^\times)^2$, so a class τ may be represented by an element $d \in k^\times$, up to squares. If Y has the form (2.1), then the twisted torsor Y^τ is given explicitly as

$$V\big(dQ_i(s,t,u) - \widetilde{Q}_i(x,y,z): \quad i = 1,2,3\big). \tag{2.4}$$

Over $\mathbb{Q}$, a class τ is represented uniquely by a squarefree integer d; we write $Y_{\mathbf{a}}^{(d)}$ instead of $Y_{\mathbf{a}}^{\tau}$.

3. Absence of $\mathbb{Q}$-points

Lemma 3.1. *Let* $\mathbf{a} \in \mathbb{Z}^3_{>0}$ *satisfy conditions* (1) *and* (2)*. If d is a squarefree integer divisible by a prime p different from* 2 *and* 5*, then* $Y_{\mathbf{a}}^{(d)}(\mathbb{Z}/p^2\mathbb{Z}) = \emptyset$.

Proof. Let $(s : t : u : x : y : z)$ be a primitive $\mathbb{Z}/p^2\mathbb{Z}$-point of $Y_{\mathbf{a}}^{(d)}$. The equations of $Y_{\mathbf{a}}^{(d)}$ imply that

$$xy + 5z^2, \quad (x+y)(x+2y), \quad \text{and} \quad ax^2 + by^2 + cz^2 \tag{3.1}$$

are all congruent to 0 modulo p. Let us first assume that $x \equiv -y \pmod{p}$. Substituting this congruence into the first and third quadrics of (3.1) we obtain

$$\begin{aligned} y^2 &\equiv 5z^2 \pmod{p}, \\ (a+b)y^2 &\equiv -cz^2 \pmod{p}. \end{aligned}$$

These congruences are either linearly independent, in which case $z \equiv y \equiv x \equiv 0 \pmod{p}$, or else $p \mid (5a+5b+c)$. In the latter case, by condition (1) we know that 5 is not a square modulo any prime p dividing $5a+5b+c$, so the congruences again imply that $z \equiv y \equiv x \equiv 0 \pmod{p}$. The defining equations of $Y_{\mathbf{a}}^{(d)}$ then imply that $s \equiv t \equiv u \equiv 0 \pmod{p}$, a contradiction. The argument for the case $x \equiv -2y \pmod{p}$ is similar. □

Proposition 3.2. *Let* $\mathbf{a} \in \mathbb{Z}^3_{>0}$ *satisfy conditions* (1)–(4)*. Then*

$$X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{et,Br}} = \emptyset.$$

Proof. By definition of the étale-Brauer set, it suffices to show that

$$Y_{\mathbf{a}}^{(d)}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset \tag{3.2}$$

for all squarefree integers d (including 1). Lemma 3.1 establishes (3.2) for all such d except those in $\langle -1, 2, 5\rangle$. Since a, b and c are positive, it is easy to see that $Y_{\mathbf{a}}^{(d)}(\mathbb{R}) = \emptyset$ for any negative d. Condition (3) implies that $Y_{\mathbf{a}}^{(d)}(\mathbb{Q}_3) = \emptyset$ for all $d \equiv 2 \pmod{3}$ and condition (4) implies that $Y^{(10)}(\mathbb{Q}_5) = \emptyset$. Thus, the only remaining case is $d = 1$. Birch and Swinnerton-Dyer prove that the del Pezzo surface $S \subset \mathbb{P}^4$ of degree 4 given by

$$xy = s^2 - 5z^2, \tag{3.3}$$

$$(x+y)(x+2y) = s^2 - 5t^2, \tag{3.4}$$

has no adelic points orthogonal to the quaternion Azumaya algebra

$$\mathcal{A} := \left(5, \frac{x+y}{x}\right) \in \operatorname{im}\left(\operatorname{Br} S \to \operatorname{Br} \mathbf{k}(S)\right);$$

see [BSD75]. The existence of a map $h\colon Y_{\mathbf{a}} \to S$ and functoriality of the evaluation pairing imply that $Y_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{h^*(\mathcal{A})} = \emptyset$. $\square$

4. Picard Groups

We compute explicit presentations for the groups $\operatorname{Pic} \overline{X}_{\mathbf{a}}$, $\operatorname{Num} \overline{X}_{\mathbf{a}}$, and $\operatorname{Pic} \overline{Y}_{\mathbf{a}}$, and thus compute their Galois cohomology. In §4.1, we describe how to obtain genus 1 fibrations on $\overline{Y}_{\mathbf{a}}$ and $\overline{X}_{\mathbf{a}}$. In §4.2, we prove that the fibers of these fibrations give us a finite index subgroup of $\operatorname{Pic} \overline{Y}_{\mathbf{a}}$, and in §4.3, we explain how to compute its saturation in $\operatorname{Pic} \overline{Y}_{\mathbf{a}}$. In §4.4, we give explicit generators for $\operatorname{Num} \overline{X}_{\mathbf{a}}$ in terms of the fibrations. Finally, in §4.5, we use the explicit computations from the previous sections to determine the low degree Galois cohomology of $\operatorname{Pic} \overline{Y}_{\mathbf{a}}, \operatorname{Pic} \overline{X}_{\mathbf{a}}$, and $\operatorname{Num} \overline{X}_{\mathbf{a}}$.

4.1. **Genus 1 fibrations.** Throughout this section, we work over an algebraically closed field.

Let Y be a $K3$ surface of degree 8, given as a closed subscheme of $\mathbb{P}^5$ by the vanishing of a net of quadrics. Let Z be the sextic curve in $\mathbb{P}^2$ that parametrizes the degeneracy locus of this net. Expanding on [Bea96, Example IX.4.5], we explain how an isolated singular point $P \in Z$ gives rise to two distinct genus 1 fibrations on Y.

Let Q be the quadric corresponding to P in the net defining Y. Then Q has rank 4, and there are two rulings on it, each realizing $V(Q)$ as a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$. Restricting to Y, we obtain two maps $\phi_P, \phi'_P\colon Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ whose respective general fibers are smooth complete intersections of two quadrics in $\mathbb{P}^3$, i.e., smooth genus 1 curves. We write F_P (resp. G_P) for the class in $\operatorname{Pic} Y$ of a fiber in ϕ_P (resp. ϕ'_P).

For the family of K3 surfaces given by $\overline{Y}_{\mathbf{a}}$, the curve $Z \subset \mathbb{P}^2$ is the union of 4 lines, each defined over $\mathbb{Q}$, and a conic. If $2a \neq b$, then the conic is geometrically irreducible and there are exactly 14 distinct singular points $P_1, \dots, P_{14}$ on Z. These points give rise to the 28 classes $F_i := F_{P_i}, G_i := G_{P_i}, i = 1, \dots, 14$ in $\operatorname{Pic} \overline{Y}_{\mathbf{a}}$, as above. The class $F_i + G_i$ is equivalent to a hyperplane section for all i, and we have

$$F_i^2 = G_i^2 = 0, \quad F_i \cdot G_i = 4, \quad F_i \cdot G_j = F_i \cdot F_j = G_i \cdot G_j = 2 \quad \text{for all } i \neq j.$$

These relations imply that $G_1, F_1, F_2, \dots, F_{14}$ generate a rank 15 sublattice of $\operatorname{Pic} \overline{Y}_{\mathbf{a}}$. We have listed equations for representatives of these classes and the Galois action of $G_{\mathbb{Q}}$ on them in Appendix A. We will use this information in later sections.

4.1.1. *Fibrations that descend to* $\overline{X}_{\mathbf{a}}$. We work in characteristic zero for the remainder of this section. Let $f\colon Y \to X$ be the K3 double-cover of a generic Enriques surface X; assume that Y is of the form (2.1). In this case, the degeneracy locus Z contains at least one singular point P; let $\phi_P, \phi'_P\colon Y \to \mathbb{P}^1$ be the fibrations described above. If there is an involution $\iota\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi \circ \sigma = \iota \circ \phi$, then each fibration descends to X, i.e., there is a genus

1 fibration $\phi_X \colon X \to \mathbb{P}^1$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "f"'] \arrow[d, "\phi_P"] & Y/\sigma = X \arrow[d, "\phi_X"] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^1/\iota = \mathbb{P}^1 \end{tikzcd}$$

commutes, and similarly for ϕ'_P.

Careful inspection of the singular locus of Z allows us to determine which points P have an associated involution ι as above. Indeed, since the quadrics defining Y are differences of a quadric in $k[x,y,z]$ and a quadric in $k[s,t,u]$, the sextic curve defining $Z \subset \mathbb{P}^2$ is the union of two (possibly reducible) cubic curves. Generically, these two cubics intersect in nine distinct points. It is precisely these singular points of Z that have an associated involution ι such that $\phi \circ \sigma = \iota \circ \phi$.

The map $\mathbb{P}^1 \to \mathbb{P}^1/\iota$ is ramified above the fixed points of ι. Since $f \colon Y \to X$ is unramified, the morphism $\phi_X \colon X \to \mathbb{P}^1$ must have non-reduced fibers above the fixed points of ι. We denote by C_P and $\widetilde{C}_P$ (resp. D_P and $\widetilde{D}_P$) the reduced subschemes of the nonreduced fibers F_P (resp. G_P) of ϕ_P.

Let us specialize to our particular Enriques surface $\overline{X}_{\mathbf{a}}$ and its K3 double cover $\overline{Y}_{\mathbf{a}}$. We already know that Z contains 14 singular points $P_1, \ldots, P_{14}$. We may renumber these points so that $P_1, \ldots, P_9$ correspond to the fibrations that descend to $\overline{X}_{\mathbf{a}}$. This gives us 36 curves $C_i, \widetilde{C}_i, D_i, \widetilde{D}_i, i = 1, \ldots, 9$ on $\overline{X}_{\mathbf{a}}$. After possibly interchanging D_i and $\widetilde{D}_i$ for some i, we have the linear equivalence relations

$$C_i + D_i = \widetilde{C}_j + \widetilde{D}_j, \quad 2(C_i - \widetilde{C}_i) = 2(D_i - \widetilde{D}_i) = 0, \quad f^*C_i = f^*\widetilde{C}_i = F_i, \quad f^*D_i = f^*\widetilde{D}_i = G_i,$$

for all i, j. Combining the projection formula with the intersection numbers on $\overline{Y}_{\mathbf{a}}$, we obtain

$$C_i^2 = D_i^2 = 0, \quad C_i \cdot D_j = C_i \cdot C_j = D_i \cdot D_j = 1, \quad C_i \cdot D_i = 2, \qquad \text{for all } i \neq j.$$

We have listed the action of the Galois group $G_{\mathbb{Q}}$ on C_i, D_i in Appendix A.

4.1.2. *Splitting field of the genus* 1 *fibrations.* Let a, b and c be indeterminates and $\mathbf{a} = (a,b,c)$; consider $Y_{\mathbf{a}}$ and $X_{\mathbf{a}}$ as surfaces over $\mathbb{Q}(a,b,c)$. The splitting field K of the fibers of all the genus 1 fibrations is a degree 2^{18} extension (explicit generators can be found in Appendix A). We can consider K as a 2^{18}-cover of $\mathbb{A}^3$. We say that the triplet $\mathbf{a_0} := (a_0, b_0, c_0) \in \mathbb{Z}^3_{>0}$ is Galois general if the special fiber $K_{(a_0,b_0,c_0)}$ is a field, i.e., if the splitting field of the fibers of the genus 1 fibrations of $Y_{\mathbf{a_0}}$ is a degree 2^{18} extension over $\mathbb{Q}$.

4.2. **Upper bounds for $\rho(Y_{\mathbf{a}})$ (after van Luijk).** Let $p \in \mathbb{Z}$ be a prime of good reduction for $Y_{\mathbf{a}}$, and write $Y_{\mathbf{a},p}$ for the mod p reduction of $Y_{\mathbf{a}}$. Suppose that $Y_{\mathbf{a}}$ has good reduction at two distinct primes p_1 and p_2, that $\rho(Y_{\mathbf{a},p_i}) \leq n$ for each i, and that the discriminants of the Picard groups of the reductions lie in different square classes. Then $\rho(Y_{\mathbf{a}}) \leq n-1$ (see [vL07, Proof of Theorem 3.1]).

Let $\ell \neq p$ be a prime, and write $\psi_p(T)$ for the characteristic polynomial of the action of Frobenius on $\mathrm{H}^2_{\text{et}}\left(Y_{\mathbf{a},p}, \mathbb{Q}_\ell\right)$. Then $\rho(Y_{\mathbf{a},p})$ is bounded above by the number of roots of ψ_p (counted with multiplicity) that are of the form $p\zeta$, where ζ is a root of unity [vL07, Corollary 2.3]. Using the Lefschetz trace formula and Newton's identities, we may compute the coefficient of T^i in $\psi_p(T)$ in terms of $\#Y_{\mathbf{a},p}(\mathbb{F}_p),\dots,\#Y_{\mathbf{a},p}(\mathbb{F}_{p^{22-i}})$. The functional equation

$$p^{22}\psi_p(T) = \pm T^{22}\psi_p(p^2/T), \tag{4.1}$$

then allows us to compute the coefficient of T^{23-i}, up to sign. In addition, for a subgroup $M \subseteq \operatorname{Pic}\overline{Y}_{\mathbf{a},p}$, the polynomial $\psi_M(T/p)$ divides $\psi_p(T)$, where $\psi_M(T)$ is the characteristic polynomial of Frobenius acting on M. Thus, knowing the action of Frobenius on a rank r subgroup of $\operatorname{Pic}\overline{Y}_{\mathbf{a},p}$, together with $\#Y_{\mathbf{a},p}(\mathbb{F}_p),\dots,\#Y_{\mathbf{a},p}(\mathbb{F}_{p^{\lceil(22-r)/2\rceil}})$, allows us to compute up to two possible $\psi_p(T)$'s, each corresponding to a choice of sign in (4.1). In some cases, this is enough information to rule out one of the sign choices; for more ways to determine the sign choice, see [EJ10].

Proposition 4.1. *Let $\mathbf{a} \in \mathbb{Z}^3_{>0}$ satisfy conditions* (5) *and* (6)*. Then $\rho(Y_{\mathbf{a}}) \leq 15$.*

Proof. Let M_p denote the subgroup of $\operatorname{Pic}\overline{Y}_{\mathbf{a},p}$ generated by $G_1, F_1, \dots, F_{14}$. Let $\widetilde{\psi}_p(T) := p^{-22}\psi_p(pT)$, so that the number of roots of $\widetilde{\psi}_p(T)$ that are roots of unity gives an upper bound for the geometric Picard number of the reduced surface. As described above, computing the action of Frobenius on M_p and computing $\#Y_{\mathbf{a},p}(\mathbb{F}_{p^i})$ for $i := 1, 2, 3, 4$ is enough to determine $\tilde{\psi}_p(T)$ for $p = 7$ and 11:

$$\begin{aligned}
\widetilde{\psi}_7(T) &= \frac{1}{7}(T-1)^8(T+1)^8(7T^6+6T^5+9T^4+4T^3+9T^2+6T+7),\\
\widetilde{\psi}_{11}(T) &= \frac{1}{11}(T-1)^8(T+1)^4(T^2+1)^2(11T^6-2T^5+T^4+12T^3+T^2-2T+11).
\end{aligned}$$

In both cases, the roots of the degree 6 factor of $\widetilde{\psi}_p(T)$ are not integral, so they are not roots of unity. We conclude that $\rho(Y_{\mathbf{a},p}) \leq 16$ for $p = 7$ and 11.

Next, we compute the square class of the discriminant Δ of each reduced Picard lattice via the Artin-Tate conjecture, which is known to hold for K3 surfaces endowed with a genus 1 fibration (see [ASD73, Mil75]):

$$\lim_{T\to p}\frac{\psi_p(T)}{(T-p)^{\operatorname{rk}(\operatorname{Pic}Y_{\mathbf{a},p})}} = p^{21-\operatorname{rk}(\operatorname{Pic}Y_{\mathbf{a},p})}\cdot \#\operatorname{Br}(Y_{\mathbf{a},p})\cdot|\Delta|$$

Observe that $\#\operatorname{Br}(Y_{\mathbf{a},p})$ is always a square [LLR05]. Write $\big[|\Delta|\big]$ for the class of $|\Delta|$ in $(\mathbb{Q}^*)/(\mathbb{Q}^*)^2$. We compute

$$\big[|\Delta(\overline{Y}_{\mathbf{a},7})|\big] = 3 \quad\text{and}\quad \big[|\Delta(\overline{Y}_{\mathbf{a},11})|\big] = 2,$$

and thus $\rho(Y_{\mathbf{a}}) \leq 15$, completing the proof. □

4.3. **Determining** $\operatorname{Pic}\overline{Y}_{\mathbf{a}}$**.** The sublattice $L := \langle G_1, F_1, \dots, F_{14}\rangle$ of $\operatorname{Pic}\overline{Y}_{\mathbf{a}}$ has discriminant 2^{17}. In this subsection we determine its saturation in $\operatorname{Pic}\overline{Y}_{\mathbf{a}}$.

Each line ℓ on the curve Z corresponds to a pencil of quadrics; the vanishing locus of this pencil S_ℓ defines a del Pezzo surface of degree 4, embedded in a hyperplane in $\mathbb{P}^5$. The inclusion of the pencil in the net of quadrics defining $\overline{Y}_{\mathbf{a}}$ gives a morphism $\overline{Y}_{\mathbf{a}} \to S_\ell$. Recall that Z contains four lines, so we obtain four such maps. Pulling back the exceptional curves on each of the del Pezzo surfaces, we obtain 16 order 2 elements in $(\operatorname{Pic}\overline{Y}_{\mathbf{a}})/L$, generated by

$$\frac{1}{2}(F_1+F_2+F_3+F_{10}+F_{12}), \quad \frac{1}{2}(F_1+G_1+F_4+F_5+F_6+F_{10}+F_{11}), \tag{4.2}$$

$$\frac{1}{2}(F_1+F_4+F_7+F_{13}+F_{14}), \quad \frac{1}{2}(F_1+G_1+F_7+F_8+F_9+F_{11}+F_{12}). \tag{4.3}$$

We owe the idea behind the proof of the following proposition to Michael Stoll and Damiano Testa [ST10].

Proposition 4.2. *Let* $\mathbf{a} \in \mathbb{Z}^3_{>0}$ *satisfy condition* (8)*. The sublattice* $L' \subseteq \operatorname{Pic} \overline{Y}_{\mathbf{a}}$ *spanned by* $G_1, F_1, \dots, F_{14}$ *and the classes in* (4.2) *and* (4.3) *is saturated.*

Proof. The lattice L' has discriminant 2^9, so it suffices to show that the induced map

$$\phi\colon L'/2L' \to \operatorname{Pic} \overline{Y}_{\mathbf{a}}/2 \operatorname{Pic} \overline{Y}_{\mathbf{a}}$$

is injective. The action of $G_{\mathbb{Q}}$ on L' factors through a finite group G whose order divides 2^{18} (see §4.1.2). Consider the induced G-equivariant homomorphism

$$\phi^G\colon \left(L'/2L'\right)^G \to \left(\operatorname{Pic} \overline{Y}_{\mathbf{a}}/2 \operatorname{Pic} \overline{Y}_{\mathbf{a}}\right)^G,$$

and note that if $\ker\phi$ is nonzero then $(\ker\phi)^G = \ker\phi^G$ is also nonzero, because any representation of a 2-group by a nonzero $\mathbb{F}_2$-vector space has a nonzero invariant subspace [Ser77, Proposition 26]. Using condition (8), it is easy to establish that $\left(L'/2L'\right)^G$ is a 2-dimensional $\mathbb{F}_2$-vector space, spanned by the classes $v_1 := [G_1+F_{10}]$ and $v_2 := [F_2+F_3+F_{12}]$. If $\phi^G(v_1) = 0$ then $G_1+F_{10} \in 2\operatorname{Pic}\overline{Y}_{\mathbf{a}}$; however, the intersection pairing on $\operatorname{Pic}\overline{Y}_{\mathbf{a}}$ is even, and $\frac{1}{2}(G_1+F_{10}) \cdot \frac{1}{2}(G_1+F_{10}) = 1$. Hence $\phi^G(v_1) \neq 0$. Similarly, $\phi^G(v_2), \phi^G(v_1+v_2) \neq 0$, and we conclude that $\ker\phi^G = 0$. □

Corollary 4.3. *Let* $\mathbf{a} \in \mathbb{Z}^3_{>0}$ *satisfy conditions* (5)*,* (6)*, and* (8)*. Then* $L' = \operatorname{Pic}\overline{Y}_{\mathbf{a}}$*.*

Proof. By Proposition 4.1 we have $\rho(Y_{\mathbf{a}}) \leq 15$. Thus L' has full rank inside $\operatorname{Pic}\overline{Y}_{\mathbf{a}}$, and it is saturated by Proposition 4.2. □

4.4. **Determining** $\operatorname{Num}\overline{X}_{\mathbf{a}}$**.** Let $M := \langle D_1, C_1, C_2, \dots, C_9\rangle \subseteq \operatorname{Num}\overline{X}_{\mathbf{a}}$. In §4.1, using intersection numbers, we calculated that $\operatorname{rk} M = 10$, and that the discriminant of the intersection lattice is 4. Since $\operatorname{Num}\overline{X}_{\mathbf{a}}$ is a rank 10 unimodular lattice (see §2.1.2), M is an index 2 subgroup of $\operatorname{Num}\overline{X}_{\mathbf{a}}$. The following proposition shows that there are only two possible saturations of M.

Proposition 4.4. *There exists a divisor* R *on* $\overline{X}_{\mathbf{a}}$ *such that* $2R$ *is linearly equivalent to either*

$$C_1 + C_2 + \cdots + C_9 \quad \text{or} \quad D_1 + C_2 + \cdots + C_9, \tag{4.4}$$

and such that $\operatorname{Num}\overline{X}_{\mathbf{a}} = \langle R, D_1, C_1, \dots, C_9\rangle$*.*

Proof. From the discussion above, we know that $\operatorname{Num}\overline{X}_{\mathbf{a}}/M \cong \mathbb{Z}/2\mathbb{Z}$. Let R be a divisor whose class in $\operatorname{Num}\overline{X}_{\mathbf{a}}$ is not in M. Without loss of generality, we may assume that

$$2R = n_1C_1 + \cdots n_9C_9 + n_{10}D_1$$

in $\operatorname{Pic}\overline{X}_{\mathbf{a}}$, where each $n_i \in \{0,1\}$. Since R pairs integrally with D_1 and C_i for all i and $R^2 \equiv 0 \pmod 2$, we must have that $n_2 \equiv n_3 \equiv \cdots n_9 \equiv 1 \pmod 2$ and $n_1 + n_{10} \equiv 1 \pmod 2$, giving the desired result. □

4.5. **Galois cohomology.** Throughout this section, we assume that $\mathbf{a} \in \mathbb{Z}^3_{>0}$ is Galois general, and we let $K_{a,b,c}$ be the splitting field described in §4.1.2. Since $\operatorname{Pic} \overline{Y}_{\mathbf{a}}$ and $\operatorname{Num} \overline{X}_{\mathbf{a}}$ are torsion-free, to compute the Galois cohomology of $\operatorname{Pic} \overline{Y}_{\mathbf{a}}$ and $\operatorname{Num} \overline{X}_{\mathbf{a}}$ it suffices to compute the action of $G_{\mathbb{Q}}$ on the curves C_i, D_i, F_i, G_i. Thus, the action of $G_{\mathbb{Q}}$ on $\operatorname{Pic} \overline{Y}_{\mathbf{a}}$ and $\operatorname{Num} \overline{X}_{\mathbf{a}}$ factors through $\operatorname{Gal}(K_{a,b,c}/\mathbb{Q})$. The action of $\operatorname{Gal}(K_{a,b,c}/\mathbb{Q})$ is described in Table A.2. Together with the generators for $\operatorname{Pic} \overline{Y}_{\mathbf{a}}$ and $\operatorname{Num} \overline{X}_{\mathbf{a}}$ given in §§4.3–4.4, it allows us to prove the following proposition.

Proposition 4.5. *Let* $\mathbf{a} \in \mathbb{Z}^3_{>0}$ *satisfy condition* (8)*. Then* $\mathrm{H}^1(G_{\mathbb{Q}}, \operatorname{Num} \overline{X}_{\mathbf{a}}) = \{1\}$. □

We used `Magma` [BCP97] for the computations involved in Proposition 4.5, but the industrious reader may verify the results by hand, using the following lemma.

Lemma 4.6. *Let G be a (pro-)finite group and A be a (continuous) torsion-free G-module. Then*

$$\mathrm{H}^1(G, A)[m] \cong \frac{(A/mA)^G}{A^G/mA^G}. \qquad \square$$

Proposition 4.5 implies that the long exact sequence in group cohomology associated to (2.2) is given by

$$0 \to \langle K_{X_{\mathbf{a}}} \rangle \to \operatorname{Pic} X_{\mathbf{a}} \to \operatorname{Num} X_{\mathbf{a}} \to \mathrm{H}^1(G_{\mathbb{Q}}, \langle K_{X_{\mathbf{a}}} \rangle) \to \mathrm{H}^1(G_{\mathbb{Q}}, \operatorname{Pic} \overline{X}_{\mathbf{a}}) \to 0.$$

The explicit description of the Galois action given in Table A.2 allows us to verify that $\operatorname{Pic} X_{\mathbf{a}} = K_{X_{\mathbf{a}}} \oplus \mathbb{Z} \cdot [C_1 + D_1]$ and that $\operatorname{Num} X_{\mathbf{a}} = \mathbb{Z} \cdot [C_1 + D_1]$. The following proposition now follows easily.

Proposition 4.7. *Let* $\mathbf{a} \in \mathbb{Z}^3_{>0}$ *satisfy condition* (8)*. Then*

$$\mathrm{H}^1(G_{\mathbb{Q}}, \operatorname{Pic} \overline{X}_{\mathbf{a}}) \cong \mathrm{H}^1(G_{\mathbb{Q}}, \langle K_{X_{\mathbf{a}}} \rangle). \qquad \square$$

Remark 4.8. Table A.2 also allows us to show that if $\mathbf{a}$ satisfies conditions (5), (6) and (8) then $\mathrm{H}^1(G_{\mathbb{Q}}, \operatorname{Pic} \overline{Y}_{\mathbf{a}}) \cong \mathbb{Z}/2\mathbb{Z}$. While this result is not logically necessary for our purposes, it is worth noting that it implies that the algebra $h^*(\mathcal{A})$ in the proof of Proposition 3.2 gives a representative of the nontrivial class of this cohomology group, and thus, up to adjustment by constant Azumaya algebras arising from the base field, $h^*(\mathcal{A})$ is the *only* algebra that can give an algebraic Brauer-Manin obstruction to the Hasse principle on $Y_{\mathbf{a}}$.

5. The Brauer-Manin obstruction

Proposition 5.1. *Let* $\mathbf{a} \in \mathbb{Z}^3_{>0}$ *satisfy conditions* (7) *and* (8)*. Then* $X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} \neq \emptyset$.

Proof. Since $\mathbf{a}$ satisfies condition (8), by Proposition 4.7 we know that $\mathrm{H}^1(G_{\mathbb{Q}}, \operatorname{Pic} \overline{X}_{\mathbf{a}}) \cong \mathrm{H}^1(G_{\mathbb{Q}}, \langle K_{X_{\mathbf{a}}} \rangle)$. Using this in conjunction with [Sko01, Theorem 6.1.2] and the isomorphism coming from the Hochschild-Serre spectral sequence

$$\mathrm{Br}_1 X / \operatorname{im} \mathrm{Br}\, \mathbb{Q} \xrightarrow{\sim} \mathrm{H}^1(G_{\mathbb{Q}}, \operatorname{Pic} \overline{X}),$$

we obtain the following partition of the algebraic Brauer set

$$X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} = \bigcup_{\tau \in \mathrm{H}^1(G_{\mathbb{Q}}, \langle K_{X_{\mathbf{a}}} \rangle)} f^{\tau}\left(Y^{\tau}_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})\right). \tag{5.1}$$

By condition (7) we have $Y_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$, and thus $X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} \neq \emptyset$. □

Proposition 5.2. *Let* $\mathbf{a} \in \mathbb{Z}^3_{>0}$ *satisfy conditions* (5)–(8), *and assume that* $X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$. *Then there exists an element* $\mathcal{A} \in \operatorname{Br} Y \setminus \operatorname{Br}_1 Y$ *such that* $Y_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} = \emptyset$.

Proof. Since $\operatorname{Br} \overline{X} \cong \mathbb{Z}/2\mathbb{Z}$ for *any* Enriques surface X [HS05, p. 3223], there is at most one nontrivial class in $\operatorname{Br} X / \operatorname{Br}_1 X$. If $\operatorname{Br} X_{\mathbf{a}} = \operatorname{Br}_1 X_{\mathbf{a}}$, then $X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset$ (by Proposition 5.1), contradicting our hypotheses. Thus we may assume that there exists an element $\mathcal{A}' \in \operatorname{Br} X_{\mathbf{a}} \setminus \operatorname{Br}_1 X_{\mathbf{a}}$. Then $X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} \cap X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}'}$, and (5.1), together with functoriality of the Brauer group imply that

$$X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \bigcup_{\tau \in \mathrm{H}^1(G_{\mathbb{Q}}, \langle K_{X_{\mathbf{a}}} \rangle)} f^{\tau}\left(Y^{\tau}_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{(f^{\tau})^* \mathcal{A}'} \right).$$

By assumption, we have $X_{\mathbf{a}}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$, so it suffices to show that $f^*\mathcal{A}' \notin \operatorname{Br}_1 Y_{\mathbf{a}}$, because then we can take $\mathcal{A} = f^*\mathcal{A}'$. Equivalently, we show that the map $f^* \colon \operatorname{Br} \overline{X}_{\mathbf{a}} \to \operatorname{Br} \overline{Y}_{\mathbf{a}}$ is injective, using Beauville's criterion [Bea09, Cor. 5.7]: f^* is injective if and only if there is no divisor class $D \in \operatorname{Pic} \overline{Y}_{\mathbf{a}}$ such that $D^2 \equiv 2 \pmod 4$ and $\sigma(D) = -D$. This is a straightforward computation, using the integral basis of $\operatorname{Pic} \overline{Y}_{\mathbf{a}}$ given in Corollary 4.3. $\square$

6. Proof of Theorems 1.2 and 1.1

Lemma 6.1. $Y_{(12,111,13)}(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$.

Proof. By the Weil conjectures, if $p \geq 22$ is a prime of good reduction, then $Y_{(12,111,13)}$ has $\mathbb{F}_p$-points. Hensel's lemma then implies that $Y_{(12,111,13)}(\mathbb{Q}_p) \neq \emptyset$ for all such primes. It remains to consider the infinite place, the primes less than 22, and the primes of bad reduction: $2, 3, 5, 13, 37, 59, 151, 157, 179, 821, 881$ and 1433. It is easy to check that $Y_{(12,111,13)}(\mathbb{R}) \neq \emptyset$, and that for a finite prime $p > 5$ in our list, $Y_{(12,111,13)}$ has a smooth $\mathbb{F}_p$-point. For the remaining three primes we exhibit explicit local points:

$$\begin{aligned}
(\sqrt{129} : 2\sqrt{21/5} : \sqrt{2113} : 1 : 4 : 5) &\in Y_{(12,111,13)}(\mathbb{Q}_2), \\
(0 : 0 : \sqrt{821/5} : -2 : 1 : \sqrt{2/5}) &\in Y_{(12,111,13)}(\mathbb{Q}_3), \\
(1 : 2\sqrt{-1} : \sqrt{136} : 1 : -4 : 1) &\in Y_{(12,111,13)}(\mathbb{Q}_5).
\end{aligned}$$
$\square$

Proof of Theorem 1.2. Combining Propositions 3.2 and 5.1 we obtain the first statement. The second statement follows from Proposition 5.2. $\square$

Proof of Theorem 1.1. The triplet $\mathbf{a} = (12, 111, 13)$ satisfies the eight required conditions: (1) – (6) and (8) are easily checked, and (7) follows from Lemma 6.1. $\square$

6.1. **More possibilities.** Although we have shown only that the triplet $\mathbf{a} = (12, 111, 13)$ satisfies conditions (1)–(8), there are in fact many triplets that do the job. Additionally, we can weaken conditions (5) and (6), and thus find even more triplets.

We used conditions (5) and (6) to ensure that $\rho(Y_{\mathbf{a},7}) = \rho(Y_{\mathbf{a},11}) = 16$ and that the discriminants of the respective intersection lattices did not differ by a square. Testing all possible isomorphism classes of $Y_{\mathbf{a}}$ modulo 7 and modulo 11, we found 10 380 possible congruence classes of $\mathbf{a}$ modulo 77 that have the desired property concerning Picard numbers and discriminants.

A computer search shows that there are 202 triplets $\mathbf{a} \in \mathbb{Z}^3_{>0}$ satisfying conditions (1)–(4) and (7) and these weaker versions of (5) and (6) with $a + b + c \leq 500$. We do not know how to verify condition (8) efficiently on a computer. However, Hilbert's irreducibility theorem says we should expect condition (8) to hold "almost always", so we feel this data does give some evidence that several triplets $\mathbf{a}$ satisfying conditions (1)–(8) exist.

6.2. **Varying the del Pezzo surface.** The Enriques surface $X_{\mathbf{a}}$ was built up from the degree 4 del Pezzo surface S given by (3.3) and (3.4). The equality $\mathrm{H}^1(G_{\mathbb{Q}}, \operatorname{Num} \overline{X}_{\mathbf{a}}) = \{1\}$ depends on the interaction of σ with S in a way that is perhaps not obvious in the proof. More precisely, consider the following K3 surfaces

$$Y_1: \begin{array}{rl} xy &= z^2 - 5s^2 \\ (x+y)(x+2y) &= z^2 - 5t^2 \\ Q_1(x,y,z) &= \widetilde{Q}_1(s,t,u) \end{array} \qquad Y_2: \begin{array}{rl} xy &= s^2 - 5u^2 \\ (x+y)(x+2y) &= s^2 - 5t^2 \\ Q_2(x,y,z) &= \widetilde{Q}_2(s,t,u) \end{array}$$

where $Q_i, \widetilde{Q}_i$ are any quadrics such that $\sigma|_{Y_i}$ has no fixed points, and let $X_i := Y_i/\sigma, i = 1, 2$, be the corresponding Enriques surfaces. Note that each Y_i maps to a del Pezzo surface which is $\mathbb{Q}$-isomorphic to S. In this case $\#\,\mathrm{H}^1(G_{\mathbb{Q}}, \operatorname{Num} \overline{X}_i) \geq 2$ regardless of the choice of $Q_i, \widetilde{Q}_i$. In fact, the quaternion algebras $\left(\frac{5z^2}{s^2}, \frac{x+y}{x}\right) \in \operatorname{Br} \mathbf{k}(X_1)$ and $\left(5, \frac{x+y}{x}\right) \in \operatorname{Br} \mathbf{k}(X_2)$ lift to non-trivial elements in $\operatorname{Br} X_1$ and $\operatorname{Br} X_2$, respectively.

Remark 6.2. Although $\frac{z^2}{s^2}$ is a square in $\mathbf{k}(Y_2)$, it is *not* a square in $\mathbf{k}(X_2)$, so $\left(\frac{5z^2}{s^2}, \frac{x+y}{x}\right) \in \operatorname{Br} \mathbf{k}(X_1)$ is not necessarily equal to $\left(5, \frac{x+y}{x}\right) \in \operatorname{Br} \mathbf{k}(X_1)$.

Remark 6.3. One can check that, up to a $\mathbb{Q}$-automorphism of $\mathbb{P}^5$ that commutes with σ, Y_1 and Y_2 are the only K3 surfaces that arise as double covers of some Enriques surface and that map to a del Pezzo surface which is $\mathbb{Q}$-isomorphic, but not equal, to S.

Appendix A. The splitting field of fibers of genus 1 fibrations

The splitting field K of the genus 1 curves $C_i, \widetilde{C}_i, \widetilde{D}_i, F_i, G_i$ is generated by

$$i, \sqrt{2}, \sqrt{5}, \sqrt{a}, \sqrt{c}, \sqrt{c^2 - 100ab}, \gamma := \sqrt{-c^2 - 5bc - 10ac - 25ab}, \tag{A.1}$$

$$\sqrt[4]{ab}, \sqrt{-2 + 2\sqrt{2}}, \sqrt{-c - 10\sqrt{ab}}, \tag{A.2}$$

$$\theta_0 := \sqrt{4a^2 + b^2},\ \delta_1 := \sqrt{a + b + c/5},\ \delta_2 := \sqrt{a + b/4 + c/10}, \tag{A.3}$$

$$\theta_1^+ := \sqrt{20a^2 - 10ab - 2bc + (10a + 2c)\theta_0},\ \theta_2^+ := \sqrt{-5a - 5/2b - 5/2\theta_0}, \tag{A.4}$$

$$\xi_1^+ := \sqrt{20a + 10b + 3c + 20\delta_1\delta_2},\ \xi_2^+ := \sqrt{4a + 2b + 2/5c + 4\delta_1\delta_2}. \tag{A.5}$$

The field extension $K/\mathbb{Q}$ is Galois, as the following relations show:

$$\sqrt{-2 - 2\sqrt{2}} = \frac{2i}{\sqrt{-2 + 2\sqrt{2}}}, \quad \sqrt{-c + 10\sqrt{ab}} = \frac{\sqrt{c^2 - 100ab}}{\sqrt{-c - 10\sqrt{ab}}}$$

$$\theta_1^- := \sqrt{20a^2 - 10ab - 2bc + (10a + 2c)\theta_0} = \frac{4a\gamma}{\theta_1^+}, \quad \theta_2^- := \sqrt{-5a - 5/2b + 5/2\theta_0} = \frac{5\sqrt{ab}}{\theta_2^+}$$

$$\xi_1^- := \sqrt{20a + 10b + 3c - 20\delta_1\delta_2} = \frac{\sqrt{c^2 - 100ab}}{\xi_1^+}, \quad \xi_2^- := \sqrt{4a + 2b + 2/5c - 4\delta_1\delta_2} = \frac{2\gamma}{5\xi_2^+}$$

Table A.1 lists two linear forms ℓ_1, ℓ_2 next to each divisor class. On the K3 surface, the vanishing of these linear forms defines a curve representing the corresponding class. On the Enriques surface, the curve $f(\overline{Y}_{\mathbf{a}} \cap V(\ell_1, \ell_2))_{\mathrm{red}}$ represents the given divisor class.

$\overline{Y}_{\mathbf{a}}$	$\overline{X}_{\mathbf{a}}$	Defining equations	$\overline{Y}_{\mathbf{a}}$	$\overline{X}_{\mathbf{a}}$	Defining equations
F_1	C_1	$s-\sqrt{5}t,\quad x+2y$	F_4	C_4	$\sqrt{c}s-\sqrt{5}u,\quad 10ax-(c+\sqrt{c^2-100ab})y$
	$\widetilde{C}_1$	$s+\sqrt{5}t,\quad x+y$		$\widetilde{C}_4$	$\sqrt{c}s+\sqrt{5}u,\quad 10ax-(c-\sqrt{c^2-100ab})y$
G_1	D_1	$s-\sqrt{5}t,\quad x+y$	G_4	D_4	$\sqrt{c}s-\sqrt{5}u,\quad 10ax-(c-\sqrt{c^2-100ab})y$
	$\widetilde{D}_1$	$s+\sqrt{5}t,\quad x+2y$		$\widetilde{D}_4$	$\sqrt{c}s+\sqrt{5}u,\quad 10ax-(c+\sqrt{c^2-100ab})y$
F_2	C_2	$\sqrt{-2+2\sqrt{2}}s-\sqrt{5}t,\quad x+\sqrt{2}y+\sqrt{5}(1-\sqrt{2})z$	F_5	C_5	$i\sqrt{2}\sqrt[4]{ab}s-u,\quad \sqrt{a}x+\sqrt{b}y+\sqrt{-c-10\sqrt{ab}}z$
	$\widetilde{C}_2$	$\sqrt{-2+2\sqrt{2}}s+\sqrt{5}t,\quad x+\sqrt{2}y-\sqrt{5}(1-\sqrt{2})z$		$\widetilde{C}_5$	$i\sqrt{2}\sqrt[4]{ab}s+u,\quad \sqrt{a}x+\sqrt{b}y-\sqrt{-c-10\sqrt{ab}}z$
G_2	D_2	$\sqrt{-2+2\sqrt{2}}s-\sqrt{5}t,\quad x+\sqrt{2}y-\sqrt{5}(1-\sqrt{2})z$	G_5	D_5	$i\sqrt{2}\sqrt[4]{ab}s-u,\quad \sqrt{a}x+\sqrt{b}y-\sqrt{-c-10\sqrt{ab}}z$
	$\widetilde{D}_2$	$\sqrt{-2+2\sqrt{2}}s+\sqrt{5}t,\quad x+\sqrt{2}y+\sqrt{5}(1-\sqrt{2})z$		$\widetilde{D}_5$	$i\sqrt{2}\sqrt[4]{ab}s+u,\quad \sqrt{a}x+\sqrt{b}y+\sqrt{-c-10\sqrt{ab}}z$
F_3	C_3	$\sqrt{-2-2\sqrt{2}}s-\sqrt{5}t,\quad x-\sqrt{2}y+\sqrt{5}(1+\sqrt{2})z$	F_6	C_6	$\sqrt{2}\sqrt[4]{ab}s+u,\quad \sqrt{a}x-\sqrt{b}y+\sqrt{-c+10\sqrt{ab}}z$
	$\widetilde{C}_3$	$\sqrt{-2-2\sqrt{2}}s+\sqrt{5}t,\quad x-\sqrt{2}y-\sqrt{5}(1+\sqrt{2})z$		$\widetilde{C}_6$	$\sqrt{2}\sqrt[4]{ab}s-u,\quad \sqrt{a}x-\sqrt{b}y-\sqrt{-c+10\sqrt{ab}}z$
G_3	D_3	$\sqrt{-2-2\sqrt{2}}s-\sqrt{5}t,\quad x-\sqrt{2}y-\sqrt{5}(1+\sqrt{2})z$	G_6	D_6	$\sqrt{2}\sqrt[4]{ab}s+u,\quad \sqrt{a}x-\sqrt{b}y-\sqrt{-c+10\sqrt{ab}}z$
	$\widetilde{D}_3$	$\sqrt{-2-2\sqrt{2}}s+\sqrt{5}t,\quad x-\sqrt{2}y+\sqrt{5}(1+\sqrt{2})z$		$\widetilde{D}_6$	$\sqrt{2}\sqrt[4]{ab}s-u,\quad \sqrt{a}x-\sqrt{b}y+\sqrt{-c+10\sqrt{ab}}z$

$\overline{Y}_{\mathbf{a}}$	$\overline{X}_{\mathbf{a}}$	Defining equations	$\overline{Y}_{\mathbf{a}}$	$\overline{X}_{\mathbf{a}}$	Defining equations
F_7	C_7	$\sqrt{c}t-u,\quad (5a+c)x+(c+\gamma)y$	F_{10}		$s-\sqrt{5}z,\quad x$
	$\widetilde{C}_7$	$\sqrt{c}t+u,\quad (5a+c)x+(c-\gamma)y$	G_{10}		$s-\sqrt{5}z,\quad y$
G_7	D_7	$\sqrt{c}t-u,\quad (5a+c)x+(c-\gamma)y$	F_{11}		$u-\sqrt{c}z,\quad \sqrt{a}x+i\sqrt{b}y$
	$\widetilde{D}_7$	$\sqrt{c}t+u,\quad (5a+c)x+(c+\gamma)y$	G_{11}		$u-\sqrt{c}z,\quad \sqrt{a}x-i\sqrt{b}y$
F_8	C_8	$\theta_2^+t-u,\quad x+(2a+b+\theta_0)/(b+\theta_0)y-\theta_1^+/(2a)z$	F_{12}		$t-z,\quad x+(1-i)y$
	$\widetilde{C}_8$	$\theta_2^+t+u,\quad x+(2a+b+\theta_0)/(b+\theta_0)y+\theta_1^+/(2a)z$	G_{12}		$t-z,\quad x+(1+i)y$
G_8	D_8	$\theta_2^+t-u,\quad x+(2a+b+\theta_0)/(b+\theta_0)y+\theta_1^+/(2a)z$	F_{13}		$\xi_2^+s-\xi_1^+t,\quad (\delta_1+2\delta_2)x+(2\delta_1+2\delta_2)y-u$
	$\widetilde{D}_8$	$\theta_2^+t+u,\quad x+(2a+b+\theta_0)/(b+\theta_0)y-\theta_1^+/(2a)z$	G_{13}		$\xi_2^+s-\xi_1^+t,\quad (\delta_1+2\delta_2)x+(2\delta_1+2\delta_2)y+u$
F_9	C_9	$\theta_2^-t-u,\quad x+(2a+b-\theta_0)/(b-\theta_0)y-\theta_1^-/(2a)z$	F_{14}		$\xi_2^-s-\xi_1^-t,\quad (\delta_1-2\delta_2)x+(2\delta_1-2\delta_2)y-u$
	$\widetilde{C}_9$	$\theta_2^-t+u,\quad x+(2a+b-\theta_0)/(b-\theta_0)y+\theta_1^-/(2a)z$	G_{14}		$\xi_2^-s-\xi_1^-t,\quad (\delta_1-2\delta_2)x+(2\delta_1-2\delta_2)y+u$
G_9	D_9	$\theta_2^-t-u,\quad x+(2a+b-\theta_0)/(b-\theta_0)y+\theta_1^-/(2a)z$			
	$\widetilde{D}_9$	$\theta_2^-t+u,\quad x+(2a+b-\theta_0)/(b-\theta_0)y-\theta_1^-/(2a)z$			

TABLE A.1. Defining equations of a curve representing a divisor class

Action on splitting field	Action on $\operatorname{Pic}\overline{X}_{\mathbf{a}}$	Action on $\operatorname{Pic}\overline{Y}_{\mathbf{a}}$	Action on splitting field	Action on $\operatorname{Pic}\overline{X}_{\mathbf{a}}$	Action on $\operatorname{Pic}\overline{Y}_{\mathbf{a}}$	Action on splitting field	Action on $\operatorname{Pic}\overline{X}_{\mathbf{a}}$	Action on $\operatorname{Pic}\overline{Y}_{\mathbf{a}}$
$\sqrt{5} \mapsto -\sqrt{5}$	$C_1 \mapsto D_1$ $D_1 \leftrightarrow \widetilde{C}_1$ $C_2 \mapsto \widetilde{C}_2$ $C_3 \mapsto \widetilde{C}_3$ $C_4 \mapsto \widetilde{D}_4$	$F_1 \leftrightarrow G_1$ $F_4 \mapsto G_4$ $F_{10} \mapsto G_{10}$	$i \mapsto -i$	$C_3 \mapsto \widetilde{D}_3$ $C_5 \mapsto \widetilde{D}_5$	$F_3 \mapsto G_3$ $F_5 \mapsto G_5$ $F_{11} \mapsto G_{11}$ $F_{12} \mapsto G_{12}$	$\sqrt[4]{ab} \mapsto i\sqrt[4]{ab}$ $\sqrt{-c-10\sqrt{ab}} \leftrightarrow$ $\sqrt{-c+10\sqrt{ab}}$	$C_5 \mapsto C_6$ $C_6 \mapsto \widetilde{D}_5$ $C_9 \mapsto \widetilde{D}_9$	$F_5 \mapsto F_6$ $F_6 \mapsto G_5$ $F_9 \mapsto G_9$ $F_{11} \mapsto G_{11}$
$\sqrt{c} \mapsto -\sqrt{c}$	$C_4 \mapsto \widetilde{D}_4$ $C_7 \mapsto \widetilde{D}_7$	$F_4 \mapsto G_4$ $F_7 \mapsto G_7$ $F_{11} \mapsto G_{11}$	$\gamma \mapsto -\gamma$	$C_7 \mapsto D_7$ $C_9 \mapsto D_9$	$F_7 \mapsto G_7$ $F_9 \mapsto G_9$ $F_{14} \mapsto G_{14}$	$\sqrt{c^2-100ab} \mapsto$ $-\sqrt{c^2-100ab}$	$C_4 \mapsto D_4$ $C_6 \mapsto D_6$	$F_4 \mapsto G_4$ $F_6 \mapsto G_6$ $F_{14} \mapsto G_{14}$
$\delta_1 \mapsto -\delta_1$ $\xi_i^+ \leftrightarrow \xi_i^-$		$F_{13} \mapsto G_{14}$ $F_{14} \mapsto G_{13}$	$\sqrt{a} \mapsto -\sqrt{a}$	$C_5 \mapsto D_5$ $C_6 \mapsto D_6$	$C_6 \mapsto D_6$ $F_6 \mapsto G_6$	$\sqrt{2} \mapsto -\sqrt{2}$ $\sqrt{-2+2\sqrt{2}} \leftrightarrow$ $\sqrt{-2-2\sqrt{2}}$	$C_2 \leftrightarrow C_3$ $C_5 \mapsto \widetilde{D}_5$ $C_6 \mapsto \widetilde{D}_6$	$F_2 \leftrightarrow F_3$ $F_5 \mapsto G_5$ $F_6 \mapsto G_6$
$\delta_2 \mapsto -\delta_2$ $\xi_i^+ \leftrightarrow \xi_i^-$		$F_{13} \leftrightarrow F_{14}$	$\theta_0 \mapsto -\theta_0$ $\theta_i^+ \leftrightarrow \theta_i^-$	$C_8 \leftrightarrow C_9$	$F_8 \leftrightarrow F_9$	$\sqrt{-2+2\sqrt{2}} \mapsto$ $-\sqrt{-2+2\sqrt{2}}$	$C_2 \mapsto \widetilde{D}_2$ $C_3 \mapsto \widetilde{D}_3$	$F_2 \mapsto G_2$ $F_3 \mapsto G_3$
$\xi_1^+ \mapsto -\xi_1^+$		$F_{13} \mapsto G_{13}$ $F_{14} \mapsto G_{14}$	$\theta_1^+ \mapsto -\theta_1^+$	$C_8 \mapsto D_8$ $C_9 \mapsto D_9$	$F_8 \mapsto G_8$ $F_9 \mapsto G_9$	$\sqrt{-c-10\sqrt{ab}} \mapsto$ $-\sqrt{-c-10\sqrt{ab}}$	$C_5 \mapsto D_5$ $C_6 \mapsto D_6$	$F_5 \mapsto G_5$ $F_6 \mapsto G_6$
$\xi_2^+ \mapsto -\xi_2^+$		$F_{13} \mapsto G_{13}$ $F_{14} \mapsto G_{14}$	$\theta_2^+ \mapsto -\theta_2^+$	$C_8 \mapsto \widetilde{D}_8$ $C_9 \mapsto \widetilde{D}_9$	$F_8 \mapsto G_8$ $F_9 \mapsto G_9$			

TABLE A.2. The Galois action on the fibers of the genus 1 fibrations.
The action on the splitting field is described by the action on the generators listed in A.1, A.2, A.3, A.4, A.5.
If a generator of K is not listed, then we assume that it is fixed. We use the same convention for the curves.

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