CORRIGENDUM HIGHER DIMENSIONAL ANALOGS OF CHÂTELET SURFACES

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1. Introduction

Let K/k be a cyclic extension of fields of degree n, and let $P(x) \in k[x]$ be a separable polynomial of degree dn or $dn-1$. Let X_0 be the affine norm hypersurface in $\mathbb{A}_k^{n+1}$ given by

$$\mathrm{N}_{K/k}(\vec{z}) = P(x). \tag{1.1}$$

In [VAV12, §2], we attempted to construct a smooth proper model X of X_0 extending the map $X_0 \to \mathbb{A}_k^1$ given by $(\vec{z}, x) \mapsto x$ to a map $X \to \mathbb{P}_k^1$. However, [VAV12, Proposition 2.1] is false whenever $n > 2$. The purpose of this note is to construct a variety $X = X_{K/k,P(x)}$ fibered over $\mathbb{P}_k^1 = \operatorname{Proj} k[x_0, x_1]$ such that

(1) X is a smooth proper compactification of X_0,
(2) the generic fiber is a Severi-Brauer variety, and
(3) the degenerate fibers lie over $V\left(P(x_0/x_1)x_1^{dn}\right)$ and consist of n rational varieties which are conjugate under $\mathrm{Gal}(K/k)$.

The variety $X \to \mathbb{P}^1$ can then replace the one given in [VAV12, §2], rectifying all the statements and proofs of [VAV12, §§3–5] (see §4 for details).

Remark 1.1. It is well known that there exists a smooth proper model X of X_0; see [CTHS03] for example. To the best of our knowledge, the known existence proofs are not explicit and it seems laborious to determine whether properties (2) and (3) hold for any of these models. Artin gives a construction of a partial compactification $\tilde{X} \to \mathbb{A}_k^1$, proper over $\mathbb{A}_k^1$, enjoying properties (2) and (3) [Art82]. We believe that extending Artin's construction to a full compactification would not result in a shorter amendment.

1.1. **Outline.** Our construction of a smooth compactification takes a cue from work of Kang: the generic fiber of our construction is the embedded Severi-Brauer variety in [Kan90].

In §2.1, we construct a partial compactification $Y \to \mathbb{A}_k^1$ of the variety $z_1 \cdots z_n = f(x) \neq 0$ for any polynomial $f(x)$. In §2.2, we give an explicit open covering of Y, which we use in §2.3 to prove that Y is smooth if and only if $f(x)$ is separable. We describe the geometry of the degenerate fibers of $Y \to \mathbb{A}_k^1$ in §2.4.

In §3.1, we construct a K/k-twist of Y, $X_0 \to \mathbb{A}_k^1$, and in §3.2 we give a full compactification $X \to \mathbb{P}_k^1$. Finally, in §4, we explain why $X \to \mathbb{P}_k^1$ satisfies properties (1)–(3) above.

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Notation. Throughout, we fix an integer $n > 1$. By the weight of a vector $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathbb{Z}^n$, we mean the sum $\sum_{m=0}^{n-1} v_m$; we also say $\mathbf{v}$ has length n. Let $\mathcal{V}_n$ denote the set of nonnegative integer vectors of weight n and length n. Write $\sigma\colon \mathcal{V}_n \to \mathcal{V}_n$ for the shift operator

$$\mathbf{v} = (v_0, v_1, \dots, v_{n-1}) \mapsto (v_1, v_2, \dots, v_{n-1}, v_0) =: \sigma(\mathbf{v}).$$

For any $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$ we define two nonnegative integers:

$$\mu(\mathbf{v}) := \max_{i\in(0,n]} (i - v_0 - \dots - v_{i-1}), \qquad \lambda(\mathbf{v}) := \mu(\mathbf{v}) + \mu(\sigma(\mathbf{v})) + \dots + \mu(\sigma^{n-1}(\mathbf{v})),$$

and for any integers i, j with $i \not\equiv j \bmod n$ and $v_j > 0$, we let $\mathbf{v}^{i,j} := (v_m + \delta_{i \bmod n, m} - \delta_{j \bmod n, m})_{m=0}^{n-1}$; note that $\mathbf{v}^{i,j} \in \mathcal{V}_n$, because $v_j > 0$. We collect a few straightforward relations used frequently below.

Lemma 1.2. *We have the following relations.*

1. $\mu(\sigma^s(\mathbf{v})) = \mu(\mathbf{v}) + v_0 + v_1 + \dots + v_{s-1} - s$ *for any* $\mathbf{v} \in \mathcal{V}_n$ *and for any* $s \in (0, n]$.
2. *For any integer* r *and any vectors* $\mathbf{v}_i, \mathbf{w}_i \in \mathcal{V}_n$ *with* $1 \le i \le r$, *such that* $\sum_{i=1}^r \mathbf{w}_i = \sum_{i=1}^r \mathbf{v}_i$, *we have* $\sum_{i=1}^r \lambda(\mathbf{w}_i) - \sum_{i=1}^r \lambda(\mathbf{v}_i) = n\left(\sum_{i=1}^r \mu(\mathbf{w}_i) - \sum_{i=1}^r \mu(\mathbf{v}_i)\right)$.
3. *Fix integers* $0 \le r < s < n$ *and fix* $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$ *such that* $v_r, v_s > 0$. *Then:* $0 \le \mu(\mathbf{v}^{r,s}) + \mu(\mathbf{v}^{s,r}) - 2\mu(\mathbf{v}) \le 1$, *and the first inequality is strict if and only if* $\mu(\mathbf{v}) = i - \sum_{m=0}^{i-1} v_m = j - \sum_{m=0}^{j-1} v_m$ *for some* $i \in (r, s]$ *and* $j \in (0, r] \cup (s, n]$.

Proof. Let $\mathbf{w} := \sigma^s(\mathbf{v})$, so that $w_j = v_{j+s}$ if $j < n - s$ and $w_j = v_{j+s-n}$ if $j \ge n - s$. Then $i - w_0 - w_1 - \dots - w_{i-1}$ equals

$$\begin{cases} ((i+s) - v_0 - \dots - v_{i+s-1}) + v_0 + \dots + v_{s-1} - s & \text{if } i + s \le n \\ ((i+s-n) - v_0 - \dots - v_{i+s-n-1}) + (n-s) - v_s - \dots - v_{n-1} & \text{otherwise.} \end{cases}$$

To conclude (1), note that since $\mathbf{v}$ has weight n, we have $v_0 + v_1 + \dots + v_{s-1} - s = (n - s) - v_s - v_{s+1} - \dots - v_{n-1}$.

By (1), for any vector $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$ we have

$$\lambda(\mathbf{v}) = n\mu(\mathbf{v}) + \sum_{m=0}^{n-1} \left((n-1-m)v_m - m\right).$$

Using the assumption that $\sum_{i=1}^r \mathbf{w}_i = \sum_{i=1}^r \mathbf{v}_i$, the proof of (2) is now a simple manipulation.

It remains to prove (3). Let $\mathbf{w}^- := \mathbf{v}^{r,s}$ and $\mathbf{w}^+ := \mathbf{v}^{s,r}$. Since $r < s$, by the definition of μ we have

$$\mu(\mathbf{v}) - 1 \le \mu(\mathbf{w}^-) \le \mu(\mathbf{v}) \quad \text{and} \quad \mu(\mathbf{v}) \le \mu(\mathbf{w}^+) \le \mu(\mathbf{v}) + 1.$$

Furthermore, $\mu(\mathbf{w}^-) = \mu(\mathbf{v}) - 1$ if and only if the maximum of $\{i - v_0 - \dots - v_{i-1}\}_{i\in(0,n]}$ is *only* achieved for $i \in (r, s]$. Similarly, $\mu(\mathbf{v}) = \mu(\mathbf{w}^+)$ if and only if the maximum of $\{i - v_0 - \dots - v_{i-1}\}_{i\in(0,n]}$ is *only* achieved for $i \in (0, r] \cup (s, n]$. □

The following notion is the fundamental book-keeping device in the construction of $X \to \mathbb{P}^1$.

Definition 1.3. *A vector* $\mathbf{v} = (v_0, v_1, \dots, v_{n-1}) \in \mathcal{V}_n$ *is well-spaced if*

$$v_i > 0 \Rightarrow v_{i+1} = v_{i+2} = \dots = v_{i+v_i-1} = 0 \text{ and } v_{i+v_i} > 0$$

for all $i \in [0, n)$. *Here indices are considered modulo* n.

For example, $\mathbf{v} = (0,3,0,0,2,0,4,0,0)$ is well-spaced whereas $\mathbf{w} = (0,3,0,0,2,4,0,0,0)$ is not. Note that $\sigma(\mathbf{v})$ is well-spaced if and only if $\mathbf{v}$ is well-spaced.

Lemma 1.4. *Let $\mathbf{v} \in \mathcal{V}_n$ be a well-spaced vector with $\ell+1$ nonzero entries indexed by $i_0 < \cdots < i_\ell$. Set $i_{\ell+1} := n + i_0$. Then $\mu(\mathbf{v}) = i_0$, and for any $r, s \in [0,\ell]$ and $j \in (i_r, i_{r+1})$, we have*

$$\mu(\mathbf{v}^{j,i_r}) = \mu(\mathbf{v}) - \left\lfloor \frac{j}{n} \right\rfloor, \qquad \mu(\sigma^{i_s}(\mathbf{v}^{j,i_r})) = 0,$$

$$\text{and if } \ell \neq 0, \qquad \mu(\mathbf{v}^{i_r,i_{r+1}}) = \mu(\mathbf{v}) + \left\lfloor \frac{i_{r+1}}{n} \right\rfloor, \quad \mu(\sigma^{i_s}(\mathbf{v}^{i_r,i_{r+1}})) = \delta_{s,r+1 \bmod \ell+1}.$$

Proof. For any vector $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$, the maximum of $\{i - v_0 - \cdots - v_{i-1}\}_{i \in (0,n]}$ is never achieved at an $i = j$ where $v_j = 0$. Additionally, since $\mathbf{v}$ is well-spaced, $v_{i_r} = i_{r+1} - i_r$ for all $r \in [0,\ell]$. Hence

$$\mu(\mathbf{v}) = \max_{r\in[0,\ell]} \left(i_r - v_{i_0} - \cdots - v_{i_{r-1}}\right) = \max_{r\in[0,\ell]} \left(i_r - (i_1 - i_0) - \cdots - (i_r - i_{r-1})\right) = i_0$$

and the maximum of $\{i - v_0 - \cdots - v_{i-1}\}_{i\in(0,n]}$ is achieved at $i = i_r$ for all $r \in [0,\ell]$.

Thus, the formulas for $\mu(\mathbf{v}^{j,i_r})$ and $\mu(\mathbf{v}^{i_r,i_{r+1}})$ follow from the same argument as in Lemma 1.2 (3). Let $\mathbf{w} := \mathbf{v}^{j,i_r}$. Then

$$\begin{aligned}
\mu(\sigma^{i_s}(\mathbf{w})) &= \mu(\mathbf{w}) + w_0 + \cdots + w_{i_s-1} - i_s && \text{by Lemma 1.2 (1),}\\
&= \mu(\mathbf{v}) - \left\lfloor \frac{j}{n} \right\rfloor + w_0 + \cdots + w_{i_s-1} - i_s && \text{by the formula for } \mu(\mathbf{v}^{j,i_r}),\\
&= \mu(\mathbf{v}) - \left\lfloor \frac{j}{n} \right\rfloor + v_0 + \cdots + v_{i_s-1} - i_s + \left\lfloor \frac{j}{n} \right\rfloor && \text{by the definition of } \mathbf{v}^{j,i_r},\\
&= \mu(\mathbf{v}) - i_0 && \text{since } v_{i_r} = i_{r+1} - i_r \text{ for } r \in [0,\ell].
\end{aligned}$$

Similarly, if $\mathbf{w} := \mathbf{v}^{i_r,i_{r+1}}$, we have

$$\begin{aligned}
\mu(\sigma^{i_s}(\mathbf{w})) &= \mu(\mathbf{w}) + w_0 + \cdots + w_{i_s-1} - i_s\\
&= \mu(\mathbf{v}) + \left\lfloor \frac{i_{r+1}}{n} \right\rfloor + w_0 + \cdots + w_{i_s-1} - i_s\\
&= \mu(\mathbf{v}) + \left\lfloor \frac{i_{r+1}}{n} \right\rfloor + v_0 + \cdots + v_{i_s-1} - i_s + \delta_{s,r+1 \bmod \ell+1} - \left\lfloor \frac{i_{r+1}}{n} \right\rfloor\\
&= \mu(\mathbf{v}) - i_0 + \delta_{s,r+1 \bmod \ell+1}.
\end{aligned}$$

□

Let $N = \binom{2n-1}{n} - 1$, and fix coordinates on $\mathbb{P}^N_k = \operatorname{Proj} k\left[\{y_\mathbf{v} : \mathbf{v} \in \mathcal{V}_n\}\right]$, and set $\mathbb{P}^N_{k[x]} := \mathbb{P}^N_k \times_{\operatorname{Spec} k} \operatorname{Spec} k[x]$ and $\mathbb{P}^N_{k(x)} := \mathbb{P}^N_k \times_{\operatorname{Spec} k} \operatorname{Spec} k(x)$.

Given a ring R and an element $a \in R$, we use the standard notation $D(a)$ to denote the open affine subscheme of $\operatorname{Spec} R$ given by $\operatorname{Spec} R_a$ and $D_+(a)$ to denote the open affine subscheme of $\operatorname{Proj} R$ given by $\operatorname{Spec}(R_a)_0$.

2. The auxiliary bundle $Y \to \mathbb{A}^1_k$

2.1. **Construction of $Y_{f(x)}$.** For any polynomial $f(x) \in k[x]$, we consider the embedding

$$\iota_{f(x)}\colon\ \operatorname{Proj} k[x][t_0,\ldots,t_{n-1}] \cap D(f(x)) \hookrightarrow \mathbb{P}^N_{k[x]} \times_{\mathbb{A}^1_k} D(f(x))$$

induced by the map $y_{\mathbf{v}} \mapsto f(x)^{\mu(\mathbf{v})} t_0^{v_0} t_1^{v_1} \cdots t_{n-1}^{v_{n-1}}$.(This is easily seen to be an embedding since it is the composition of the (n)-uple embedding (see [Har77, p. 13]) with a scaling of the coordinates by an appropriate power of $f(x)$.) The image of $\iota_{f(x)}$ is cut out by the equations:

$$f(x)^{\sum_{i=1}^{r} \mu(\mathbf{w}_i)} \prod_{i=1}^{r} y_{\mathbf{v}_i} = f(x)^{\sum_{i=1}^{r} \mu(\mathbf{v}_i)} \prod_{i=1}^{r} y_{\mathbf{w}_i} \tag{2.1}$$

for all integers r and all sets of vectors $\mathbf{w}_i, \mathbf{v}_i$, with $1 \leq i \leq r$, such that $\sum_{i=1}^{r} \mathbf{w}_i = \sum_{i=1}^{r} \mathbf{v}_i$. Let $Y_{f(x)}$ be the closure in $\mathbb{P}^N_{k[x]}$ of the image of $\iota_{f(x)}$.

Lemma 2.1. *The order n automorphism $\phi\colon \mathbb{P}^N_{k[x]} \to \mathbb{P}^N_{k[x]}$, $y_{\mathbf{v}} \mapsto y_{\sigma(\mathbf{v})}$ preserves $Y_{f(x)}$.*

Proof. Fix an integer r and vectors $\mathbf{v}_i, \mathbf{w}_i$, with $1 \leq i \leq r$, such that $\sum_{i=1}^{r} \mathbf{v}_i = \sum_{i=1}^{r} \mathbf{w}_i$. It is clear that $\sum_{i=1}^{r} \sigma(\mathbf{v}_i) = \sum_{i=1}^{r} \sigma(\mathbf{w}_i)$. Moreover, by Lemma 1.2(1),

$$\sum_{i=1}^{r} \mu(\sigma(\mathbf{v}_i)) - \sum_{i=1}^{r} \mu(\sigma(\mathbf{w}_i)) = \sum_{i=1}^{r} \mu(\mathbf{v}_i) - \sum_{i=1}^{r} \mu(\mathbf{w}_i).$$

Therefore, ϕ preserves the relations (2.1). □

2.2. An open covering.

Proposition 2.2. *The open subvarieties $\{D_+(y_{\mathbf{v}}) \subseteq \mathbb{P}^N_{k[x]} : \mathbf{v} \in \mathcal{V}_n \text{ well-spaced}\}$ cover $Y_{f(x)}$.*

Proof. Let $\mathbf{w} = (w_0, w_1, \ldots, w_{n-1}) \in \mathcal{V}_n$ be a vector that is not well-spaced. We will show that $D_+(y_{\mathbf{w}}) \cap Y_{f(x)} \subset D_+(y_{\mathbf{v}})$ for some well-spaced vector $\mathbf{v} \in \mathcal{V}_n$. Let $i_0 < \cdots < i_\ell$ be the indices such that $w_{i_j} > 0$, and set $i_{\ell+1} = i_0 + n$.

If $\mu(\mathbf{w}) > i_0 \geq 0$, then there exists an $r \in [0, \ell)$ such that $\mu(\mathbf{w}) = i_{r+1} - w_{i_0} - w_{i_1} - \cdots - w_{i_r}$. Fix the smallest such r; then $i_{r+1} - i_r - w_{i_r} > 0$. Since $\mathbf{w}$ has length n and weight n, there exists an $r < s \leq \ell$ such that $(i_{s+1} - i_s - w_{i_s}) < 0$; fix the largest such s. Then by our choice of r and s, if $\mu(\mathbf{w}) = j - v_0 - \cdots - v_{j-1}$, we must have $j \in (i_r, i_s]$. Therefore by Lemma 1.2(3), the defining equations for $Y_{f(x)}$ include the relation

$$y_{\mathbf{w}}^2 = y_{\mathbf{w}^{i_r,i_s}} y_{\mathbf{w}^{i_s,i_r}},$$

so $D_+(y_{\mathbf{w}}) \subset D_+(y_{\mathbf{w}^{i_r,i_s}})$. After possibly repeating the argument we may assume that $\mu(\mathbf{w}) = i_0$.

If $i_{r+1} - i_r - w_{i_r} = 0$ for all r, then $\mathbf{w}$ is well-spaced. Otherwise, fix the smallest integer r such that $|i_{r+1} - i_r - w_{i_r}| > 0$; since $\mu(\mathbf{w}) = i_0$, we must have $i_{r+1} - i_r - w_{i_r} < 0$. Since $\mathbf{w}$ has length n and weight n, there exists an $r < s \leq \ell$ such that $(i_{s+1} - i_s - w_{i_s}) > 0$; fix the smallest such s. Now by our choice of r and s, if $\mu(\mathbf{w}) = j - v_0 - \cdots - v_{j-1}$, we must have $j \in [0, i_r] \cup (i_s, n)$. Then by the same argument as above, the defining equations for $Y_{f(x)}$ include the relation

$$y_{\mathbf{w}}^2 = y_{\mathbf{w}^{i_r,i_s}} y_{\mathbf{w}^{i_s,i_r}}.$$

By replacing $\mathbf{w}$ with $\mathbf{w}^{i_s,i_r}$, we reduce the value of $|i_{r+1} - i_r - w_{i_r}|$. Repeating this process we will arrive at a well-spaced vector $\mathbf{v}$ in finitely many steps. □

2.3. **Smoothness of $Y_{f(x)}$.**

Proposition 2.3. *Let $\mathbb{A}^n_{k[x]} = \operatorname{Spec} k[x][Z_0, \dots, Z_{n-1}]$. Let $\mathbf{v} \in \mathcal{V}_n$ be a well-spaced vector with $\ell+1$ nonzero entries indexed by $i_0 < \cdots < i_\ell$ and set $i_{\ell+1} := i_0 + n$. Then the map*

$$\frac{y_{\mathbf{w}}}{y_{\mathbf{v}}} \mapsto \left(\prod_{j\in[0,n),\, v_j=0} Z_j^{w_j} \right) \times \left(\prod_{r=0}^{\ell} Z_{i_r}^{\mu(\sigma^{i_{r+1}}(\mathbf{w}))} \right)$$

yields an isomorphism $Y_{f(x)} \cap D_+(y_{\mathbf{v}}) \cong V(Z_{i_0} \cdots Z_{i_\ell} - f(x)) \subset \mathbb{A}^n_{k[x]}$. In particular, $Y_{f(x)}$ is a compactification of the variety in $\mathbb{A}^n_{k[x]}$ given by $Z_0 Z_1 \cdots Z_{n-1} = f(x)$.

Corollary 2.4. *The variety $Y_{f(x)}$ is smooth if and only if $f(x)$ is a separable polynomial.*

Proof. This follows from Propositions 2.2 and 2.3, and the Jacobian criterion. □

Proof of Proposition 2.3. Set $i_{-1} = i_\ell - n$. The proof of the proposition differs slightly in the case when $\ell = 0$. To give a unified presentation, if $\ell = 0$ then we set $y_{\mathbf{v}^{i_0,i_{\ell+1}}} := f(x) y_{\mathbf{v}}$. Consider the following functions on $Y_{f(x)} \cap D_+(y_{\mathbf{v}})$ for $j = i_0, i_0+1, \dots, i_0+n-1$:

$$g_j := \begin{cases} y_{\mathbf{v}^{j,i_r}} y_{\mathbf{v}}^{-1} & \text{if } i_r < j < i_{r+1} \text{ for some } 0 \le r \le \ell, \\ y_{\mathbf{v}^{i_r,i_{r+1}}} y_{\mathbf{v}}^{-1} & \text{if } j = i_r, 0 \le r \le \ell. \end{cases} \tag{2.2}$$

Lemma 1.4 shows that the map sends $g_j \to Z_{j \bmod n}$. Thus, to prove the map is a well-defined isomorphism, we will show that

$$y_{\mathbf{w}} y_{\mathbf{v}}^m = \left(\prod_{j\in[0,n),\, v_j=0} y_{\mathbf{v}^{j,i_r}}^{w_j} \right) \times \left(\prod_{r=0}^{\ell} y_{\mathbf{v}^{i_r,i_{r+1}}}^{\mu(\sigma^{i_{r+1}}(\mathbf{w}))} \right)$$

where $m = -1 + \sum_{j, v_j=0} w_j + \sum_{r=0}^{\ell} \mu(\sigma^{i_{r+1}}(\mathbf{w}))$. By Lemmas 1.4 and 1.2(1), we have

$$\begin{aligned} \sum_{\substack{j\in[0,n),\\ v_j=0}} w_j \cdot \mu(\mathbf{v}^{j,i_r}) + \sum_{r=0}^{\ell} \mu(\sigma^{i_{r+1}}(\mathbf{w}))\mu(\mathbf{v}^{i_r,i_{r+1}}) &= (m+1)\mu(\mathbf{v}) - \sum_{j=0}^{i_0-1} w_j + \mu(\sigma^{i_{\ell+1}}(\mathbf{w})) \\ &= (m+1)\mu(\mathbf{v}) + \mu(\mathbf{w}) - i_0 = m\mu(\mathbf{v}) + \mu(\mathbf{w}). \end{aligned}$$

Hence, by (2.1), it suffices to prove that $\mathbf{w} + m\mathbf{v}$ is equal to

$$\mathbf{w}' := \sum_{j\in[0,n),\, v_j=0} w_j \mathbf{v}^{j,i_r} + \sum_{r=0}^{\ell} \mu(\sigma^{i_{r+1}}(\mathbf{w}))\mathbf{v}^{i_r,i_{r+1}}.$$

For j such that $v_j = 0$, it is evident that $w'_j = w_j + m v_j$. Further,

$$\begin{aligned} w'_{i_s} &= (m+1)v_{i_s} - \sum_{j=i_s+1}^{i_{s+1}-1} w_j + \mu(\sigma^{i_{s+1}}(\mathbf{w})) - \mu(\sigma^{i_s}(\mathbf{w})) \\ &= (m+1)v_{i_s} - \sum_{j=i_s+1}^{i_{s+1}-1} w_j + \sum_{j=i_s}^{i_{s+1}-1} w_j - i_{s+1} + i_s && \text{by Lemma 1.2(1)} \\ &= m v_{i_s} + w_{i_s} && \text{since } v_{i_s} = i_{s+1} - i_s. \end{aligned}$$

□

2.4. The degenerate fibers of $Y_{f(x)} \to \mathbb{A}^1_k$.

Proposition 2.5. *Let $Q \in V(f(x)) \in \mathbb{A}^1_k$ be a closed point. The fiber $Y_{f(x),Q}$ consists of n rational $(n-1)$-dimensional irreducible components which are permuted cyclically by the automorphism ϕ of Lemma 2.1.*

Proof. For $i = 0, \dots, n-1$, we define $S_i := Y_{f(x),Q} \cap V(\langle y_{\mathbf{w}} : \mu(\sigma^{i+1}(\mathbf{w})) > 0 \rangle)$. From the definition, it follows that ϕ acts on the set $\{S_i : 0 \le i \le n-1\}$ via the permutation

$$S_0 \mapsto S_{n-1} \mapsto S_{n-2} \mapsto \ldots \mapsto S_1 \mapsto S_0.$$

We claim that $Y_{f(x),Q} = S_0 \cup S_1 \cup \cdots \cup S_{n-1}$ and that each S_i is an irreducible rational $(n-1)$-dimensional variety.

Let $\mathbf{v} \in \mathcal{V}_n$ be a well-spaced vector and let $i_0 < \cdots < i_\ell$ be the indices of the nonzero entries of $\mathbf{v}$. By Proposition 2.3, $Y_{f(x),Q} \cap D_+(y_{\mathbf{v}})$ is isomorphic to a union of $\ell+1$ hyperplanes in $\mathbb{A}^n_{\mathbf{k}(Q)} = \operatorname{Spec} \mathbf{k}(Q)[Z_0, \dots, Z_{n-1}]$. Furthermore, the hyperplane $Z_{i_r} = 0$ is isomorphic to the subvariety

$$V\left(\left\langle \frac{y_{\mathbf{w}}}{y_{\mathbf{v}}} : \mu(\sigma^{i_r+1}(\mathbf{w})) > 0 \right\rangle\right) \subset Y_{f(x),Q} \cap D_+(y_{\mathbf{v}}).$$

Hence, $Y_{f(x),Q} \cap D_+(y_{\mathbf{v}})$ is a dense open subset of $S_{i_0} \cup S_{i_1} \cup \cdots \cup S_{i_\ell}$. Since the open subvarieties $\{D_+(y_{\mathbf{v}}) \cap Y_{f(x),Q} : \mathbf{v} \text{ well-spaced}\}$ cover $Y_{f(x),Q}$, this completes the proof. □

3. The construction of $X_{K/k,P(x)}$

Let K/k be a cyclic Galois extension of degree n. Fix a basis $\{\alpha_0, \dots, \alpha_{n-1}\}$ of K as a k-vector space, as well as a generator τ of $\operatorname{Gal}(K/k)$.

3.1. A K/k twist of $Y_{f(x)}$.

Let T be a set of representatives for the orbits of $\mathcal{V}_n$under the action of the shift operator σ. Consider the K-isomorphism $\psi\colon \operatorname{Proj} K[\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \to \operatorname{Proj} K[\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}]$ determined by

$$y_{\mathbf{v}} \mapsto \alpha_0 z_{\mathbf{v}} + \alpha_1 z_{\sigma(\mathbf{v})} + \cdots + \alpha_{n-1} z_{\sigma^{n-1}(\mathbf{v})} \quad \text{for } \mathbf{v} \in T,$$
$$y_{\sigma^i(\mathbf{v})} \mapsto \sum_{j=0}^{n-1} \tau^i(\alpha_j) z_{\sigma^j(\mathbf{v})} \quad \text{for } \mathbf{v} \in T \text{ and } i = 1, \dots, n-1.$$

Define $X^0_{K,f(x)} := \psi^{-1}_{K[x]}(Y_{f(x)})$. Abusing notation, we write τ for the endomorphism of $\operatorname{Proj} k[x][\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \times_{k[x]} K[x]$ given by $\operatorname{id} \times \tau$. Let $\phi\colon \mathbb{P}^N_{k[x]} \to \mathbb{P}^N_{k[x]}$ be the automorphism of Lemma 2.1. The following diagram commutes

$$\begin{tikzcd} \operatorname{Proj} K[x][\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \arrow[r, "\psi_{K[x]}"] \arrow[d, "\tau"'] & \operatorname{Proj} K[x][\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \arrow[d, "\phi_{K[x]}"] \\ \operatorname{Proj} K[x][\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \arrow[r, "\psi_{K[x]}"] & \operatorname{Proj} K[x][\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \end{tikzcd} \tag{3.1}$$

By Lemma 2.1, the map ϕ preserves $Y_{f(x)}$. Together with the commutativity of the above diagram, this implies that $X^0_{K,f(x)}$ descends to a $k[x]$-scheme.

3.2. **Compactifying $X^0_{K,f(x)}$.**

Proposition 3.1. *Let $\tilde{P}(x_0, x_1) \in k[x_0, x_1]$ be a squarefree homogeneous polynomial of degree dn, for some positive integer d. There exists a smooth projective variety $X = X_{K/k,P(x)} \to \mathbb{P}^1_k$ such that $X_{\mathbb{A}^1} \cong X^0_{K,\tilde{P}(x,1)}$ and that $X_{\mathbb{P}^1\setminus\{0\}} \cong X^0_{K,\tilde{P}(1,x')}$, where $x' = 1/x$.*

Proof. We will construct X by glueing $Y_{\tilde{P}(x,1)}$ and $Y_{\tilde{P}(1,x')}$ over $\operatorname{Spec} k[x, x^{-1}]$ and $\operatorname{Spec} k[x', x'^{-1}]$, in a way which is compatible with the map ψ from §3.1. Let $y_{\mathbf{v}}$ denote the coordinates on $Y_{\tilde{P}(x,1)}$ and let $y'_{\mathbf{v}}$ denote the coordinates on $Y_{\tilde{P}(1,x')}$. By Lemma 1.2(2), the morphism

$$Y_{P(1,x')} \times_{\mathbb{A}^1} \operatorname{Spec} k[x', x'^{-1}] \to Y_{P(x,1)} \times_{\mathbb{A}^1} \operatorname{Spec} k[x, x^{-1}]$$

which sends $y_{\mathbf{v}} \mapsto (x')^{d\lambda(\mathbf{v})} y'_{\mathbf{v}}$ and $x \mapsto 1/x'$ is well-defined and is an isomorphism. Since $\lambda(\mathbf{v}) = \lambda(\sigma(\mathbf{v}))$, this morphism is compatible with ψ and thus gives a glueing of $X_{K,\tilde{P}(x,1)}$ and $X_{K,\tilde{P}(1,x')}$. $\square$

4. A smooth proper model of X_0

Let K/k be a cyclic extension of degree n and let $P(x) \in k[x]$ be a separable polynomial of degree dn or $dn - 1$ for some d. Let $\tilde{P}(x_0, x_1) = P(x_0/x_1)x_1^{dn}$, and let $X \to \mathbb{P}^1_k$ be the smooth projective variety of Proposition 3.1.

Proposition 4.1. *The variety X is a smooth proper compactification of X_0, the generic fiber of $X \to \mathbb{P}^1$ is a Severi-Brauer variety, and the degenerate fibers of $X \to \mathbb{P}^1$ lie over the roots $\tilde{P}$ and consist of the union of n rational varieties all conjugate under* $\operatorname{Gal}(K/k)$*.*

Proof. The compatibility (3.1) together with Proposition 2.3 and Corollary 2.4 implies that

$$(X \times_{\mathbb{P}^1} \mathbb{A}^1) \cap D_+(z_{(1,1,\ldots,1)}) \cong X_0,$$

which gives the first claim. The second claim is immediate from the construction of X, and the third claim follows from Proposition 2.5 and the compatibility (3.1). $\square$

Replacing the variety X of [VAV12, §2] with the variety X constructed here rectifies all the statements and proofs of [VAV12, §§3–5], with the exception of part of the proof of Theorem 1.3. Precisely, we must correct the construction of the Châtelet p-fold bundle given by $u^p P_\infty(x) + P_0(x)$ in the proof of Theorem 1.3. To do so, we carry out the same construction as above over the polynomial rings $k[u, x]$, $k[1/u, x]$,$k[u, 1/x]$,$k[1/u, 1/x]$ and glue to construct a bundle over $\mathbb{P}^1 \times \mathbb{P}^1$.

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