# Arithmetic of K3 surfaces: open problems

Unless otherwise specified, $X$ is an algebraic K3 surface and $k$ is a number field.

(1) (McKinnon) Given $X/k$ and $P \in X(k)$, determine if there is a non-constant map $f\colon \mathbb{P}^1_k \to X$ such that $P$ is in the image.
(Beauville) Same question over $\overline{k}$.
(Colliot-Thélène) Related conjecture by Beilinson: $\mathrm{CH}_0(\overline{X})$ should be $\mathbb{Z}$.

(2) Given $X/k$, compute the rank of $\operatorname{Pic} X$. Same question for $\operatorname{Pic} \overline{X}$.
(Poonen) If one can determine $\operatorname{Pic} \overline{X}$, then one can determine $\operatorname{Pic} X$.

(3) Given $L \in \operatorname{Pic} X$ and $m > 1$, determine if $L = mM$ for some $M \in \operatorname{Pic} X$. Given a nonzero $L$, determine an upper bound for $\{m : L \in m \operatorname{Pic} X\}$.
(Baragar) Given $\Lambda \subset \operatorname{Pic} X$, decide if $\Lambda = \operatorname{Pic} X$. Determine if $\Lambda$ is saturated in $\operatorname{Pic} X$. This is equivalent to determining if $\Lambda$ is saturated in $H^2(X, \mathbb{Z})$.

(4) (McKinnon) Given $L \in \operatorname{Pic} X$, decide if $L = [C]$ for some integral curve $C$.

(5) (Poonen) Given $D \in \operatorname{Div} X$, can one compute $H^0(X, \mathcal{O}(D))$? The answer seems to be yes.

(6) (Silverman) Suppose that $X$ is defined over $\mathbb{C}$ and $\operatorname{Aut}(X)$ is infinite. Let $C$ be an integral curve on $X$ and $P \in X$. Suppose that $(\operatorname{Aut}(X) \cdot P) \cap C$ is infinite. Does this imply that there exists a non-trivial automorphism $\sigma$ of infinite order mapping $C$ to itself? This is an analogue of Mordell-Lang for K3 surfaces.
(Beauville) If $C$ is an integral curve on $X/\mathbb{C}$ and $\sigma$ is an automorphism of infinite order such that $\sigma C = C$, then $g(C) \leq 1$.

(7) Let $\sigma$ be an automorphism of infinite order of $X$ and let $C \subset X$ be a curve. Assume that the set of periodic points of $\sigma$ contained in $C$ is infinite. Does this imply that $C$ is periodic, i.e. $\sigma^n C = C$ for some $n$?

(8) (Skorobogatov) What does the Bloch-Beilinson conjecture say explicitly for a K3 over $\mathbb{Q}$? What interesting numbers should appear in special values of $L$-functions?

(9) (Bogomolov and Tschinkel) Given $X/\overline{\mathbb{F}}_p$, is there a rational curve through every $\overline{\mathbb{F}}_p$-point? Same for $\overline{\mathbb{Q}}$? There is a positive result due to Bogomolov and Tschinkel for the first question for Kummer surfaces over $\overline{\mathbb{F}}_p$.
(McKinnon) Over $\overline{\mathbb{Q}}$, this is equivalent to the K3 being "rationally connected" over $\overline{\mathbb{Q}}$.

(10) (Skorobogatov) Find a formula for

$$\# \frac{\operatorname{Br} A}{\operatorname{Br}_1 A} = \# \operatorname{im}(\operatorname{Br} A \to \operatorname{Br} \overline{A})$$

for an abelian surface $A$. This is finite (proved by Skorobogatov and Zarhin). Same question for K3s. Given a fixed $A$ (or $X$), the size can be made arbitrarily large by making a suitable extension of the base field.

(11) (Wittenberg) Is there a K3 surface $X$ over a number field $k$ such that

$$\frac{\operatorname{Br} X}{\operatorname{Br} k} = 0?$$

(12) (McKinnon) Find $X/k$ with geometric Picard rank 1 and an accumulating curve over $k$, i.e., 100% of $k$-points of $X$ lie on the curve (asymptotically, ordered by height).

There are examples of K3s with accumulating curves, but all have large geometric Picard rank.

(13) (Silverman) Find an example of a K3 surface with an interesting automorphism of infinite order (e.g., excluding composition of involutions).

(14) (Poonen) Describe the possibilities for $\operatorname{Aut}(X/\mathbb{C})$ as an abstract group. Sterk proved that $\operatorname{Aut}(X/\mathbb{C})$ is finitely generated.
(Kumar) Can one compute generators for $\operatorname{Aut}(X)$? Is there an upper bound on the number of generators?
(Poonen) Can one compute the relations?
(Bender) Can one compute the minimum number of generators?
(Silverman) Is there an upper bound on the number of generators of the subgroup of $\operatorname{Aut}(X)$ generated by all the involutions?

(15) (Poonen) Given K3 surfaces $X$ and $Y$ over $\overline{\mathbb{Q}}$, can one decide if $X \cong Y$?
(McKinnon) Given K3 surfaces $X$ and $Y$ over $\overline{\mathbb{Q}}$, can one decide if $\operatorname{Aut} X \cong \operatorname{Aut} Y$ as abstract groups?

(16) Is there a K3 surface $X$ over a number field $k$ such that $X(\mathbb{A}_k) \neq \emptyset$ and $X$ satisfies weak approximation? If so, find it.
(Colliot-Thélène) Is there a K3 surface $X$ over a number field $k$ such that $X(k) \neq \emptyset$ and $X$ satisfies weak weak approximation?

(17) Is there a K3 surface $X$ over a number field $k$ with $X(k)$ nonempty and finite? Nonempty and not Zariski dense? What about over an arbitrary infinite field? Is there a K3 surface $X$ having only finitely many points over its own function field?

(18) (Poonen) Given $X/\mathbb{F}_p$ is $X(\mathbb{F}_p(t))$ finite?
(Beauville) Not always.
(Poonen) Can you compute $X(\mathbb{F}_p(t))$? Is $X(\mathbb{F}_p(t))/\operatorname{Aut}(X)$ finite?

(19) Find a K3 surface over a number field with geometric Picard rank 1 for which one can either prove that the rational points are potentially dense or prove that they are not potentially dense.

(20) (Cantat) Suppose that $X(k)$ is Zariski dense. Must there exist a finite extension $L/k$ and an embedding $L \subseteq \mathbb{C}$ such that $X(L)$ is analytically dense in $X(\mathbb{C})$?
(Cantat) This is true for Abelian varieties.

(21) (Cantat and Silverman) Suppose that $X(k)$ is Zariski dense and $v$ is a place of $k$. Must there exist a finite extension $L/k$ and a place $w$ over $v$ such that $X(L)$ is $w$-adically dense in $X(L_w)$? Must there exist a finite extension $L/k$ such that $X(L)$ is $w$-adically dense in $X(L_w)$ for all $w$ over $v$? Must there exist a finite extension $L/k$ such that $X(L)$ is dense in $X(\mathbb{A}_L)$?

(22) Is $X(k)$ always dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$, i.e., is the Brauer-Manin obstruction the only one to weak approximation? To the Hasse principle?

(23) (Cantat) Let $X$ be defined over $\mathbb{C}$. Does there exist $\mathbb{C}^2 \dashrightarrow X$ meromorphic and generically of maximal rank?

(24) (Colliot-Thélène and Ojanguren) Let $Y$ be an Enriques surface and $X$ be the associated K3 double cover. Is the map

$$\frac{\operatorname{Br} \overline{Y}}{\operatorname{Br} k} \longrightarrow \frac{\operatorname{Br} \overline{X}}{\operatorname{Br} k}$$

always injective? If not, how can one determine if it is injective in any given example?

(25) (Baragar) Find a K3 surface $X$ over a number field $k$ such there are infinitely many orbits of $k$-rational curves under the action of $\operatorname{Aut} X$.

(26) In the following all K3 surfaces $S$ are embedded in $\mathbb{P}^2 \times \mathbb{P}^2$ such that the projections do not contract any curves and $\mathcal{A}$ is a subset of the automorphism group. We are still assuming $k$ is a number field. We define

$$S[\mathcal{A}] = \{P \in S : \mathcal{A}(P) \text{ is finite}\}.$$

(a) K3 Uniform Boundedness Conjecture:
There is a constant $c = c(k)$ such that for all K3 surfaces $S/k$,

$$\#S[\mathcal{A}](k) \leq c.$$

(b) K3 Manin-Mumford Conjecture:
Let $C \subset S$ be a curve such that $\phi(C) \neq C$ for all $C \subsetneq A$. Then $C \cap S[\mathcal{A}]$ is finite.

(c) (Weak) K3 Lehmer Conjecture:
Fix $S/k$. There are constants $c = c(S/k) > 0$ and $\delta = \delta(S/k)$ so that

$$\hat{h}(P) \geq \frac{c}{[L:k]^\delta} \text{for all } L/k \text{ and } P \in S(L) \setminus S[\mathcal{A}].$$

(d) K3 Lang Height Conjecture:
There is a constant $c = c(k)$ such that for all K3 surfaces $S/k$,

$$\hat{h}(P) \geq c \cdot h(S) \quad \text{for all } P \in S(k) \setminus S[\mathcal{A}].$$

(Here $h(S)$ is the height of S as a point in the moduli space of K3 surfaces.)

(e) K3 Serre Image-of-Galois Conjecture:
For any subgroup $\mathcal{B} \subseteq \mathcal{A}$, let

$$S_{\mathcal{B}} := \left\{P \in S(\bar{k}) : \mathcal{B} \text{ is the stabilizer of } P \text{ in } \mathcal{A}\right\},$$

and define

$$\rho_{\mathcal{B}} \colon \operatorname{Gal}(k(S_{\mathcal{B}})/k) \to \operatorname{SymGp}(S_{\mathcal{B}}).$$

There is a constant $c = c(S/k)$ so that for all subgroups $\mathcal{B} \subset \mathcal{A}$ of finite index,

$$(\operatorname{SymGp}(S_{\mathcal{B}}) : \operatorname{Image}(\rho_{\mathcal{B}})) < c.$$

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