1. Let $A$ be an $11 \times 11$ matrix, and assume that the eigenspace of 0 for $A$, $E_0(A)$, is 5-dimensional. What is the rank of the $A$?

2. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be linear transformation and let $A$ be an $n \times n$ matrix such that $T(x) = Ax$. If $T$ is one-to-one, is 0 an eigenvalue of $A$?

3. If $A$ is an $n \times n$ matrix, can $A^T = A^{-1}$?

4. Can the eigenspace of the value 0 equal the column space of an $n \times n$ matrix $A$? I.e., can $E_0(A) = \text{Col}(A)$?

5. Suppose $A$ is an $m \times n$ matrix and $B$ is an $n \times m$ matrix. If $A^{-1} = B$, what is the nullity of $B$? Justify your answer.

6. Let $A$ and $B$ be $n \times n$ matrices and assume that $A$ is invertible. Then is $(A + B)$ invertible? Is $(I + BA^{-1})$ invertible?

7. Let $T_1, T_2, \ldots, T_k$ be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $T_1 \circ T_2 \circ \ldots T_k$ is onto, are $T_1, \ldots, T_k$ each onto as well?

8. Let $A$ and $B$ be $n \times n$ matrices and let $T_1$ and $T_2$ be linear transformations given by $T_1(x) = Ax$ and $T_2(x) = Bx$. If $\det(B) \det(A) = 7$, determine the relationship between $\ker(T_1 \circ T_2)$ and $\text{null}(BA)$.

9. Let $A$ and $B$ be $n \times n$ matrices. Assume that $v \in \mathbb{R}^n$ is an eigenvector for $A$ and an eigenvector for $B$. Is $v$ an eigenvector for $AB$?

10. Let $A$ and $B$ be $n \times n$ matrices. Assume that $v \in \mathbb{R}^n$ is an eigenvector for $A$ and an eigenvector for $B$. Is $v$ an eigenvector for $A^T B^T$?

11. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation such that

\[
T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -2 + 3a \end{bmatrix}, \quad \text{and} \quad T\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ -3 \end{bmatrix}.
\]

(a) Give a $2 \times 2$ matrix $A$ such that $T(x) = Ax$ for every $x \in \mathbb{R}^2$.

(b) Find a specific value of $a$ that allows every vector in $\text{range}(A)$ to also be in $\text{null}(A)$.

12. Prove each of the following statements:

(a) If $S$ is a subspace and $u_1, u_2, \ldots, u_m \in S$ are such that

\[
\text{span}(u_1, u_2, \ldots, u_m) = S,
\]

then any set of vectors $v_1, v_2, \ldots, v_k \in S$, where $k > m$, is linearly dependent.

(b) Let $A$ and $B$ be $n \times n$ matrices. If $\det(AB) \neq 0$, then $\det(A)$ and $\det(B)$ are nonzero.
13. For each of the following questions, either prove the statement, or give a counterexample:

(a) Let $A$ be an $n \times n$ matrix. Let $B = A^T$. The dimensions of the row and column spaces are the same for $A$ and $B$.

(b) Let $v_1, v_2 \in \mathbb{R}^m$, and let $A$ be an $n \times m$ matrix. If $v_1 \neq v_2$ and $Av_1 = Av_2$, then $\text{rank}(A) < m$.

(c) For any vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^m$, $S = \text{span}(v_1, v_2, \ldots, v_k)$ is a subspace of $\mathbb{R}^m$ with dimension $k$.

(d) If $A$ is a $3 \times 5$ matrix, the corresponding linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ cannot be 1-to-1.

(e) If $A$ is a $3 \times 5$ matrix, the corresponding linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ cannot be onto.

(f) If $A$ is an $n \times n$ matrix, and the vectors $v_1, v_2 \in \mathbb{R}^n$ are eigenvectors of the matrix $A$, then $v_1 + v_2$ is an eigenvector of $A$.

(g) Let $A$ be an $n \times n$ matrix with $0$ as an eigenvalue. Then $n$ may be $1$.

(h) Let $A$ be an $n \times n$ matrix with $0$ as an eigenvalue. Then the dimension of $\text{Col}(A)$ is not equal to $n$.

(i) Let $A$ be an $n \times n$ matrix with $0$ as an eigenvalue. Then the linear transformation given by $T(x) = Ax$ is not one-to-one.

(j) Let $A$ and $B$ be two matrices and assume that $B$ has one more column than $A$. Then $A$ and $B$ can both be invertible.

(k) Let $A$ be an $n \times (n-1)$ matrix and let $B$ be an $n \times n$ matrix. Assume that a column of $B$ is a scalar multiple of a column of $A$, then $\text{det}(B)$ is nonzero.

(l) Let $A$ be a $3 \times 3$ matrix with $\text{nullity}(A) = 1$. Then $\text{det}(A) > 1$.

(m) If $A$ and $B$ are $n \times n$ matrices, then $\text{det}(A + B) = \text{det}(A) + \text{det}(B)$.

(n) If $\{v_1, v_2, \ldots, v_k\}$ are linear independent vectors, then $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal set of nonzero vectors.

(o) If $U$ and $V$ are both subspaces of $\mathbb{R}^n$, then the union of $U$ and $V$ is a subspace of $\mathbb{R}^n$.

(p) If $U$ and $V$ are both subspaces of $\mathbb{R}^n$, then the intersection of $U$ and $V$ is a subspace of $\mathbb{R}^n$.

(q) Let $A$ be an $n \times n$ matrix. If $\lambda$ is an eigenvalue of $A$, then $\lambda^{-1}$ is an eigenvalue of $(A^T)^{-1}$.

(r) If $A$ is an $n \times n$ matrix with $n$ distinct nonzero eigenvalues, then $A$ has $n$ linearly independent eigenvectors.

(s) Let $B_1 = \{u_1, u_2, u_3\}$ and $B_2 = \{v_1, v_2, v_3\}$ be two bases of $\mathbb{R}^3$. Suppose that $u_1 = av_1 + bv_2 + cv_3$, $u_2 = dv_1 + ev_2 + f v_3$, and $u_3 = gv_1 + hv_2 + iv_3$, then the change of basis matrix from $B_1$ to $B_2$ is

$$
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix}.
$$