RATIONAL POINTS ON SURFACES

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The goal of these lectures is to serve as a "user's guide" to obstructions to the existence of k-points on smooth projective varieties, where k is a global field. As such, the focus will be on examples and the main ideas behind results, rather than detailed proofs (for those I will refer you to a number of excellent references).

Remark 0.1. In addition to asking whether the existence of k_v-points for all places v implies the existence of a k-point (i.e., whether the Hasse principle holds), one can also ask whether a set of k_v-points can be approximated by a k-point, i.e., whether X satisfies weak approximation. Many of the tools, conjectures, and results for the question of existence of rational points also hold for the question of density. For time and space considerations, we will focus mainly on the Hasse principle and its obstructions and mention weak approximation only when there are differences.

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General notation. Throughout, k will denote a field, k^{alg} will be a fixed algebraic closure, and k^{sep} will be the separable closure of k in k^{alg}. We write G_k for the absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$.

A k-variety will be a separated scheme of finite type over k. If X is a k-variety we will write X^{alg} and X^{sep} for the base changes $X \times_{\operatorname{Spec} k} \operatorname{Spec} k^{\mathrm{alg}}$ and $X \times_{\operatorname{Spec} k} \operatorname{Spec} k^{\mathrm{sep}}$, respectively. If k has characteristic 0, then we will sometimes write $\overline{k}$ for $k^{\mathrm{alg}} = k^{\mathrm{sep}}$ and $\overline{X}$ for $X^{\mathrm{alg}} = X^{\mathrm{sep}}$.

If X and S are k-schemes, we write $X(S) = \mathrm{Hom}_k(S, X)$. If $S = \operatorname{Spec} A$, then we write $X(A)$ for $X(\operatorname{Spec} A)$.

1. An example of a variety that fails the Hasse principle

Example 1.1 ([Lin40, Rei42])**.** Let C denote the smooth genus 1 curve given by

$$V(2y^2 - x^4 + 17z^4) \subset \mathbb{P}_{\mathbb{Q}}(1, 2, 1),$$

where $\mathbb{P}_{\mathbb{Q}}(1,2,1)$ denotes the weighted projective space in which x and z have degree 1 and y has degree 2. Alternatively, we can view C as the intersection of the following two quadrics in $\mathbb{P}^3$

$$Q_1 : 2Y^2 - W^2 + 17Z^2, \quad Q_2 : WZ - X^2.$$

We claim that $C(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ and $C(\mathbb{Q}) = \emptyset$.

Let us first prove local solubility. The existence of an $\mathbb{R}$-point is clear. The curve C has good reduction at all $p \neq 2, 17$. As the Weil conjectures (proved by Weil in the case of curves [Wei48]) show that a smooth genus 1 curve always has an $\mathbb{F}_q$-point (for any q), $C(\mathbb{F}_p) \neq \emptyset$ for all primes $p \neq 2, 17$. Furthermore, a smooth $\mathbb{F}_p$-point lifts to a $\mathbb{Q}_p$-point by Hensel's Lemma. Since $2 \equiv 6^2 \bmod 17$ and $3^4 \equiv 17 \bmod 64$, Hensel's Lemma also implies that

$$\left(\sqrt{2} : \sqrt{2} : 0\right) \in C(\mathbb{Q}_{17}) \quad \text{and} \quad \left(\sqrt[4]{17} : 0 : 1\right) \in C(\mathbb{Q}_2).$$

Now we prove that $C(\mathbb{Q}) = \emptyset$. Assume that $C(\mathbb{Q}) \neq \emptyset$ and let $(x_0 : y_0 : z_0)$ be a $\mathbb{Q}$-rational point. By scaling, we may assume that $x_0, y_0, z_0 \in \mathbb{Z}$. If there exists a prime p that divides x_0 and z_0, then p^4 must divide $2y_0^2$ and so $p^2 | y_0$. Therefore, we may scale x_0, y_0, z_0 to ensure that x_0 and z_0 are relatively prime. A similar argument shows that we may also assume that $\gcd(x_0, y_0) = \gcd(y_0, z_0) = 1$.

Since x_0, y_0 and z_0 are pairwise relatively prime, for any odd prime $p | y_0$, we must have $17 \in \mathbb{F}_p^{\times 2}$. Since 17 is 1 modulo 4, quadratic reciprocity then implies that $p \in \mathbb{F}_{17}^{\times 2}$ for any odd prime $p | y_0$. Since $-1, 2 \in \mathbb{F}_{17}^{\times 2}$, this implies that $y_0 = y_0'^2 \bmod 17$. Substituting this into the defining equation for C shows that $2 \in \mathbb{F}_{17}^{\times 4}$, resulting in a contradiction. ⋄

This proof arguably raises more questions than it answers. Was it necessary to work with the prime $p = 17$? Does a similar argument work for any equation of the form

$$ay^2 - x^4 + bz^4 = 0?$$

If an analogous argument for $ay^2 - x^4 + bz^4 = 0$ doesn't result in a contradiction, do we expect there to be a rational point?

Given these questions, one might prefer to have a proof which is less *ad hoc*, perhaps as the expense of brevity, which may then generalize more easily to other varieties.

Example 1.2. We again consider the curve C given by

$$V(2y^2 - x^4 + 17z^4) \subset \mathbb{P}_{\mathbb{Q}}(1, 2, 1),$$

and give a different argument to prove that it has no rational points. We will phrase our argument in terms of the Hilbert symbol. Let v be a place of $\mathbb{Q}$ and let $a, b \in \mathbb{Q}_v^\times$. Recall the definition of the Hilbert symbol

$$\langle a, b\rangle_v = \begin{cases} -1 & \text{if } as^2 + bt^2 = u^2 \text{ has only the trivial solution in } \mathbb{Q}_v, \\ 1 & \text{otherwise.} \end{cases}$$

Since a conic over $\mathbb{Q}_v$ has infinitely many nontrivial solutions as soon as there is one nontrivial solution, the Hilbert symbol may equivalently be defined as

$$\langle a, b\rangle_v = \begin{cases} -1 & \text{if } b \notin \mathrm{Norm}_{\mathbb{Q}_v(\sqrt{a})/\mathbb{Q}_v}(\mathbb{Q}_v(\sqrt{a})^\times) \\ 1 & \text{if } b \in \mathrm{Norm}_{\mathbb{Q}_v(\sqrt{a})/\mathbb{Q}_v}(\mathbb{Q}_v(\sqrt{a})^\times) \end{cases}$$

In this proof, we will work with the model of C that is given by the intersection of two quadrics in $\mathbb{P}^3$. Let $P = [X_0, Y_0, Z_0, W_0] \in C(\mathbb{Q}_v)$. Since $Z_0W_0 = X_0^2$, whenever $Z_0W_0Y_0 \neq 0$, we have

$$\langle 17, Z_0/Y_0\rangle_v = \langle 17, W_0/Y_0\rangle_v.$$

We claim that $\langle 17, Z_0/Y_0\rangle_v = 1$ if $v \neq 17$ and $\langle 17, Z_0/Y_0\rangle_v = -1$ if $v = 17$. Indeed, if $17 \in \mathbb{Q}_v^{\times 2}$ (which includes $v = 2$ or $v = \infty$) then this is clear. Assume that $17 \notin \mathbb{Q}_v^{\times 2}$ and $v \neq 17$. Since $v((W_0/Y_0)^2 - 17(Z_0/Y_0)^2) = v(2) = 0$, at least one of W_0/Y_0 or Z_0/Y_0 must be a v-adic unit. As $\mathbb{Q}_v(\sqrt{17})/\mathbb{Q}_v$ is unramified if $v \neq 17$, all v-adic units are norms and so $\langle 17, Z_0/Y_0\rangle_v = \langle 17, W_0/Y_0\rangle_v = 1$.

It remains to consider the case that $v = 17$. Modulo 17, $(W_0/Y_0)^2 \equiv 2$ so W_0/Y_0 is a square modulo 17 if and only if 2 is a fourth power modulo 17. Since $\mathbb{F}_{17}^{\times 4} = \{1, -1, 4, -4\}$, $W_0/Y_0 \bmod 17 \notin \mathbb{F}_{17}^{\times 2}$ and so $(17, W_0/Y_0)_{17} = -1$.

Now assume that there exists a point $P = [X_0, Y_0, Z_0, W_0] \in C(\mathbb{Q})$. One can check that $Z_0W_0Y_0 \neq 0$, thus we may consider the product of Hilbert symbols

$$\prod_v \langle 17, W_0/Y_0\rangle_v.$$

One the one hand, this product is equal to -1 by the above claim. On the other hand, since $W_0/Y_0 \in \mathbb{Q}^\times$ the product formula for Hilbert symbols states that this product is 1. This gives a contradiction, thus $C(\mathbb{Q}) = \emptyset$. ◇

At first blush, this argument may seem no better than then previous argument. However, we will see later that this argument is an example of a general phenomenon known as the Brauer-Manin obstruction.

2. The Brauer group

2.1. **The Brauer group of a field.** We will give a brief overview of the Brauer group of a field. For more details see [GS06, §§2,4] and [Mil, Chap. 4].

Let k be a field, let k^{sep} denote a separable closure, and let $G_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$. A central simple algebra (CSA) $\mathcal{A}$ over k is a finite-dimensional k-algebra whose center is k and which has no non-trivial proper 2-sided ideals. Wedderburn's theorem implies that every central simple algebra over k is isomorphic to $\mathrm{M}_n(D)$ for some positive integer n and some division algebra D with center k. Let $\mathcal{A}$ and $\mathcal{A}'$ be two central simple k-algebras; we say that $\mathcal{A}$ and $\mathcal{A}'$ are Brauer equivalent if $\mathcal{A} \otimes_k \mathrm{M}_n(k) \cong \mathcal{A}' \otimes_k \mathrm{M}_m(k)$ for some $n, m \in \mathbb{Z}_{>0}$. Then we define the Brauer group of k

$$\mathrm{Br}\, k := \frac{\{\mathrm{CSA}/k\}}{\text{Brauer equivalence}};$$

this set forms an abelian group under tensor product. Equivalently, one can define the Brauer group using Galois cohomology

$$\mathrm{Br}\, k := \mathrm{H}^2(G_k, k^{\mathrm{sep}\times}).$$

(For a proof that these definitions are, in fact, equivalent see [GS06, Thm. 4.4.5 and Cor. 2.4.10].)

Exercise 2.1. Assume that k is algebraically closed. Prove that $\mathrm{Br}\, k = 0$. (Hint: Let D be division ring that is finite dimensional over k, let $d \in D^\times$, and consider $k(d)$.) $\diamond$

Example 2.2 (Wedderburn's little theorem)**.** If k is finite, then $\mathrm{Br}\, k = 0$. $\diamond$

Exercise 2.3. Prove that $\mathrm{Br}\, \mathbb{R} \cong \mathbb{Z}/2\mathbb{Z}$ and that the unique nontrivial element is represented by the Hamiltonian quaternions

$$\mathbb{H} := \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij,$$

where $i^2 = j^2 = -1$ and $ji = -ij$. $\diamond$

Example 2.4. If k is a nonarchimedean local field, then local class field theory gives a canonical isomorphism, the invariant map,

$$\mathrm{inv}_k \colon \mathrm{Br}\, k \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

If k is the completion of a global field at a nonarchimedean place v, then we will write inv_v for inv_k. $\diamond$

Example 2.5. Let k be a global field and let Ω_k denote the set of places of k. Then the fundamental exact sequence of global class field theory completely characterizes the Brauer group of k; indeed, it states that the sequence

$$0 \longrightarrow \mathrm{Br}\, k \longrightarrow \bigoplus_{v \in \Omega_k} \mathrm{Br}\, k_v \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \tag{2.1}$$

is exact. Here inv_v denotes the invariant map from Example 2.4 in the case that v is nonarchimedean, it denotes the unique composition $\mathrm{Br}\, k_v \xrightarrow{\sim} \frac{1}{2}\mathbb{Z}/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$ in the case that v is real, and it denotes the unique inclusion $\mathrm{Br}\, \mathbb{C} = 0 \hookrightarrow \mathbb{Q}/\mathbb{Z}$ if v is complex. $\diamond$

If K is any extension of k, we have a homomorphism

$$\operatorname{Br} k \to \operatorname{Br} K, \quad \mathcal{A} \mapsto \mathcal{A} \otimes_k K.$$

We define the relative Brauer group (of the extension K/k) to be the kernel of this homomorphism and we denote it $\operatorname{Br}(K/k)$. A Brauer class $\mathcal{A} \in \operatorname{Br} k$ is said to be split by K or split over K if $\mathcal{A}$ is contained in the subgroup $\operatorname{Br}(K/k)$. If K/k is a Galois extension, then Hilbert's Theorem 90 and the inflation-restriction sequence together yield an isomorphism

$$\operatorname{Br}(K/k) \cong \mathrm{H}^2(\operatorname{Gal}(K/k), K^\times).$$

An element $\mathcal{A} \in \operatorname{Br} k$ is cyclic if there exists a finite cyclic extension L/k (i.e., a Galois extension whose Galois group is a finite cyclic group) such that $\mathcal{A}$ is split by L.

Exercise 2.6. Let L/k be a finite cyclic extension of degree n, let σ be a generator of the Galois group $\operatorname{Gal}(L/k)$, and let $b \in k^\times$. Consider the finite-dimensional k-algebra

$$(\sigma, b) := \frac{\oplus_{i=0}^{n-1} L y^i}{y^n = b, \sigma(\alpha) y = y\alpha \text{ for all } \alpha \in L}.$$

Prove that (σ, b) is a central simple k-algebra which is split by L. ⋄

Throughout, we will identify the algebras from Example 2.6 with their class in $\operatorname{Br} k$.

If L/k is a cyclic extension, then $\mathrm{H}^1(\operatorname{Gal}(L/k), L^\times)$ can be computed using Tate cohomology yielding that $\operatorname{Br}(L/k) \cong k^\times / \operatorname{Norm}_{L/k}(L^\times)$. In fact, this isomorphism can be made explicit. As explained in [GS06, Cor. 4.7.4], we have

$$\frac{k^\times}{\operatorname{Norm}_{L/k}(L^\times)} \xrightarrow{\sim} \operatorname{Br}(L/k), \quad b \mapsto (\sigma, b), \tag{2.2}$$

where σ is a fixed generator of $\operatorname{Gal}(L/k)$. In particular, the algebra (σ, b) is trivial in $\operatorname{Br} k$ if and only if $b \in \operatorname{Norm}_{L/k}(L^\times)$.

Example 2.7. Let k be a nonarchimedean local field and let L/k be an *unramified* cyclic extension. If $\sigma \in \operatorname{Gal}(L/k)$ induces the Frobenius map on the residue field then [Mil, Chap. IV, Ex. 4.2 and Prop. 4.3]

$$\operatorname{inv}_k((\sigma, b)) = \frac{v(b)}{[L:k]} \in \mathbb{Q}/\mathbb{Z}.$$

⋄

Remark 2.8. If σ and τ are generators of $\operatorname{Gal}(L/k)$, then for any $b \in k^\times$, (σ, b) and (τ, b) may not be Brauer equivalent, but always generate the same *subgroup* of $\operatorname{Br} k$. In situations where we are concerned with the subgroup generated by (σ, b) rather than the particular generator, we will abuse notation and write $(L/k, b)$ for any algebra of the form (σ, b), where σ is a generator of $\operatorname{Gal}(L/k)$.

If k contains the n^{th} roots of unity, then by Kummer theory any cyclic extension L is of the form $k(\sqrt[n]{a})$ for some $a \in k^\times$. Further, any generator $\sigma \in \operatorname{Gal}(L/k)$ is determined by a primitive n^{th} root of unity ζ. In this case, we write $(a, b)_\zeta$ for the cyclic algebra class $(\sqrt[n]{a} \mapsto \zeta\sqrt[n]{a}, b)$ and $(a, b)_n$ for $(k(\sqrt[n]{a})/k, b)$. The algebra (class) $(a, b)_\zeta$ is called a symbol algebra.

Exercise 2.9. Assume that $\mu_n \subset k$. Let $a, b \in k^\times$ and fix a primitive n^{th} root of unity ζ. Prove that $(a, b)_\zeta = (b, a)_{\zeta^{-1}}$ and $(a, b)_\zeta = (a, b(-a)^i)_\zeta$ for all $i \in \mathbb{Z}$. ⋄

2.2. **The Brauer group of a scheme.** Let X be a scheme. We define the (cohomological) Brauer group of X to be

$$\operatorname{Br} X := \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m).$$

If X is regular, integral, and noetherian, then we have an injection $\operatorname{Br} X \hookrightarrow \operatorname{Br} \mathbf{k}(X)$, where $\mathbf{k}(X)$ denotes the function field of X. Furthermore, if n is a positive integer that is invertible on X or if n is any positive integer and $\dim X \leq 2$, then we have an exact sequence

$$0 \longrightarrow (\operatorname{Br} X)[n] \longrightarrow (\operatorname{Br} \mathbf{k}(X))[n] \xrightarrow{\oplus_x \partial_x} \bigoplus_{x \in X^{(1)}} \mathrm{H}^1\left(\kappa(x), \frac{1}{n}\mathbb{Z}/\mathbb{Z}\right), \tag{2.3}$$

where $X^{(1)}$ denotes the set of codimension 1 points of X and $\kappa(x)$ denotes the residue field of x. The maps ∂_x are called residue maps and an element $\mathcal{A} \in \operatorname{Br} \mathbf{k}(X)[n]$ is said to be unramified at $x \in X^{(1)}$ if $\mathcal{A} \in \ker \partial_x$. Thus, we may view $(\operatorname{Br} X)[n]$ as the unramified subgroup of $(\operatorname{Br} \mathbf{k}(X))[n]$.

Lemma 2.10. *Let $L/\mathbf{k}(X)$ be a cyclic extension of prime degree p that is unramified at $x \in X^{(1)}$. Then $(L/\mathbf{k}(X), b) \in \ker \partial_x$ if and only if x splits completely in L or $v_x(b) \equiv 0 \bmod p$, where v_x denotes the discrete valuation associated to x.*

Remark 2.11. There are many definitions of residue maps in the literature. Although the definitions do not always agree, in each case the kernels agree. Since we are mainly concerned with the kernel of ∂_x, we will allow ourselves this ambiguity. See [CT95, Remark 3.3.2] for an overview of the different definitions.

In the case that X is a variety over a field k, we can consider the following filtration of the Brauer group:

$$\operatorname{Br}_0 X := \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X) \subset \operatorname{Br}_1 X := \ker(\operatorname{Br} X \to \operatorname{Br} X^{\mathrm{sep}}) \subset \operatorname{Br} X.$$

Elements of $\operatorname{Br}_0 X$ are constant and elements of $\operatorname{Br}_1 X$ are algebraic. Any remaining elements are termed transcendental. We say that X has trivial Brauer group if $\operatorname{Br} X = \operatorname{Br}_0 X$.

2.2.1. *Algebraic Brauer classes.* If $\mathbf{k}[X^{\mathrm{sep}}]^\times = k^{\mathrm{sep}\times}$, i.e., if the only regular invertible functions on X^{sep} are constants, then the exact sequence of low degree terms from the Hochschild-Serre spectral sequence yields

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} X^{\mathrm{sep}})^{G_k} \to \operatorname{Br} k \to \operatorname{Br}_1 X \to \mathrm{H}^1(G_k, \operatorname{Pic} X^{\mathrm{sep}}) \to \mathrm{H}^3(k, \mathbb{G}_m).$$

If k is a local or global field, then $\mathrm{H}^3(k, \mathbb{G}_m) = 0$. (This follows from statements in [NSW08] as explained in [Poo14, Rmk. 6.7.10].) Hence, if k is a local or global field then we have an isomorphism

$$\frac{\operatorname{Br}_1 X}{\operatorname{Br}_0 k} \xrightarrow{\sim} \mathrm{H}^1(k, \operatorname{Pic} X^{\mathrm{sep}}). \tag{2.4}$$

This isomorphism is the main tool in computing explicit representatives for the algebraic part of the Brauer group. See [VA08, EJ12] for some examples.

2.2.2. *Transcendental Brauer classes: general approaches.* For simplicity, we restrict to the case when $\mathrm{char}(k) = 0$. From the definition of Br_1, we have an inclusion

$$\frac{\mathrm{Br}\, X}{\mathrm{Br}_1 X} \hookrightarrow \left(\mathrm{Br}\, \overline{X}\right)^{G_k}$$

This map need not be surjective (see [HS05] for an example), but Colliot-Thélène and Skorobogatov proved that the cokernel is always finite [CTS13].

By [Gro68, pp.144-147], we have an exact sequence of abelian groups

$$0 \to (\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \to \mathrm{Br}\, \overline{X} \to \oplus_\ell \mathrm{H}^3(\overline{X}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \to 0,$$

where b_2 is the second Betti number of X and ρ is the rank of $\mathrm{NS}\, \overline{X}$. This exact sequence allows us to compute the group structure of $\mathrm{Br}\, \overline{X}$; however, we need to understand the structure of $\mathrm{Br}\, \overline{X}$ as a Galois module. To the best of my knowledge, there is currently no general method to do so.

2.2.3. *Transcendental or geometric Brauer classes: some examples.* The first example of an explicit transcendental Brauer class was constructed by Artin and Mumford in 1972 [AM72]. The first example exploited for arithmetic purposes was constructed by Harari in 1994 [Har96]. While we are still far from a general approach, over the past decade there have been a number of advances in the computation of transcendental elements. We review a few highlights here:

(1) Our understanding of the Weil-Châtelet group of elliptic curves with rational 2-torsion can be translated into techniques for constructing representatives of 2-torsion Brauer classes on certain K3 surfaces (typically Kummer surfaces of products of elliptic curves with full rational 2-torsion or those with a genus 1 fibration whose Jacobian fibration has full rational 2-torsion) [Wit04, HS05, Ier10].

(2) The purity theorem (2.3) can be used to give representatives for 2-torsion Brauer classes on surfaces that are double covers of ruled surfaces [CV14a, IOOV, BBM$^+$]. This approach can also be viewed as a generalization of (1) (see [CV14b, CV14a]). If one is only interested in the structure of $\mathrm{Br}\, \overline{X}$ as a Galois module, and not in constructing representatives, then the same results can be obtained by purely cohomological arguments [Sko].

(3) Hodge theoretic constructions yield K3 surfaces of degree 2 together with representatives for 2-torsion Brauer classes. While the constructions were originally stated over the complex numbers, they can be carried out over any field [HVAV11, HVA13].

(4) If X is the Kummer surface of a product of elliptic curves E and E', then Skorobogatov and Zarhin relate the ℓ-torsion of the Brauer group of X to homomorphisms between E and E' and between $E[\ell]$ and $E'[\ell]$ [SZ12]. This was used by Ieronymou, Skorobogatov, and Zarhin [ISZ11] to compute the odd part of the transcendental Brauer group of diagonal quartic surfaces and by Newton [New] to compute the odd part of the transcendental Brauer group of $\mathrm{Kum}(E \times E)$ where E is an elliptic curve with complex multiplication by a maximal order.

3. Local and global points

Throughout this section, we assume that k is a global field and that X is a smooth, geometrically integral k-variety. If S is a finite set of places of k, we write $\mathcal{O}_{k,S}$ for the ring of S-integers. If v is a place of k, we write k_v for the completion of k at v and $\mathcal{O}_v$ for the valuation ring in k_v. Let Ω_k denote the set of places of k and we write $\mathbb{A}_k$ for the adèle ring of k. Recall that $\mathbb{A}_k$ is the restricted product $\prod'_{v\in\Omega_k}(k_v, \mathcal{O}_v)$ and as such has a topology.

For any k-variety X, there exists a finite set $S \subset \Omega_k$ and a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_{k,S}$ such that $\mathcal{X}_k \cong X$. Then one can show that the set of adèlic points $X(\mathbb{A}_k)$ is equal to the resticted product $\prod'_{v\in\Omega_k}(X(k_v), \mathcal{X}(\mathcal{O}_v))$ [Poo14, Exer. 3.4]. The set $X(k_v)$ inherits a topology from k_v and hence $X(\mathbb{A}_k)$ can be given the restricted product topology (see [Poo14, §2.5.2] or [Con12] for more details). If X is proper, then $X(k_v) = \mathcal{X}(\mathcal{O}_v)$ and so $X(\mathbb{A}_k) = \prod_{v\in\Omega_k} X(k_v)$.

3.1. **The Brauer-Manin obstruction.** In this section we describe the Brauer-Manin obstruction, which was introduced by Manin [Man71]. Manin observed that the fundamental exact sequence of class field theory (2.1) and the functoriality of the Brauer group could be combined to define a subset $X(\mathbb{A}_k)^{\mathrm{Br}}$ of $X(\mathbb{A}_k)$ that contains $X(k)$.

For any $P_v \in X(k_v)$ and any $\mathcal{A} \in \operatorname{Br} X$, we can pullback $\mathcal{A}$ along P_v and obtain an element in $\operatorname{Br} k_v$, the evaluation of $\mathcal{A}$ in P_v; we denote this element $\mathcal{A}(P_v)$.

Lemma 3.1. *Assume that there exists a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_v$ with $\mathcal{X}_{k_v} \cong X$ and such that $\mathcal{A} \in \operatorname{im}(\operatorname{Br}\mathcal{X} \to \operatorname{Br} X)$. Then $\mathcal{A}(P_v) = 0 \in \operatorname{Br} k_v$ for all $P_v \in \mathcal{X}(\mathcal{O}_v)$.*

Proof. By the functoriality of the Brauer group, $\mathcal{A}(P_v) \in \operatorname{Br}\mathcal{O}_v$. By [Mil80, IV.2.13], the quotient map $\mathcal{O}_v \to \mathbb{F}_v$ induces an isomorphism $\operatorname{Br}\mathcal{O}_v \to \operatorname{Br}\mathbb{F}_v$. As $\operatorname{Br}\mathbb{F}_v$ is trivial, this completes the proof. □

Lemma 3.2. *For any $\mathcal{A} \in \operatorname{Br} X$, the image of $X(\mathbb{A}_k) \longrightarrow \prod_v \operatorname{Br} k_v, (P_v) \mapsto (\mathcal{A}(P_v))$ is contained in $\oplus_v \operatorname{Br} k_v$.*

Proof. For any $\mathcal{A} \in \operatorname{Br} X$ and any $(P_v) \in X(\mathbb{A}_k)$, all but finitely many places of v satisfy the assumptions of Lemma 3.1. □

Thus, for any $\mathcal{A} \in \operatorname{Br} X$, we have the following commutative diagram

$$\begin{array}{ccc} X(k) & \hookrightarrow & X(\mathbb{A}_k) \\ \downarrow{\scriptstyle \mathcal{A}(-)} & & \downarrow{\scriptstyle \mathcal{A}(-)} \\ \operatorname{Br} k & \hookrightarrow & \bigoplus_v \operatorname{Br} k_v \end{array}$$

Therefore, by the fundamental sequence of global class field theory (see Example 2.5), we have

$$X(k) \subset X(\mathbb{A}_k)^{\mathcal{A}} := \left\{ (P_v) \in X(\mathbb{A}_k) : \sum_v \operatorname{inv}_v \mathcal{A}(P_v) = 0 \in \mathbb{Q}/\mathbb{Z} \right\}.$$

For any subset $S \subset \operatorname{Br} X$, we can consider the set of adelic points orthogonal to the elements in S, which we denote by $X(\mathbb{A}_k)^S := \cap_{\mathcal{A}\in S} X(\mathbb{A}_k)^{\mathcal{A}}$. We will be particularly interested in the cases where $S = \operatorname{Br} X$ and $S = \operatorname{Br}_1 X$; these are called the Brauer-Manin set and the algebraic Brauer-Manin set of X, respectively, and denoted $X(\mathbb{A}_k)^{\mathrm{Br}}$ and $X(\mathbb{A}_k)^{\mathrm{Br}_1}$, respectively. It is clear that we have the following containments.

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)^{\mathrm{Br}_1} \subset X(\mathbb{A}_k).$$

Lemma 3.3. *Let k_v be a local field and let X be a smooth k-variety. For any $\mathcal{A} \in \operatorname{Br} X$,*

$$\operatorname{ev}_{\mathcal{A}} \colon X(k_v) \to \operatorname{Br} k_v$$

is locally constant.

Proof. To prove continuity it suffices to show that $\operatorname{ev}_{\mathcal{A}}^{-1}(\mathcal{A}')$ is open for all $\mathcal{A}' \in \operatorname{im} \operatorname{ev}_{\mathcal{A}}$. By replacing $[\mathcal{A}]$ with $[\mathcal{A}] - [\operatorname{ev}_{\mathcal{A}}(x)]$, we reduce to showing that $\operatorname{ev}_{\mathcal{A}}^{-1}(0)$ is open. We may also assume that X is irreducible and affine.

By [Hoo80], $\mathcal{A} \in \operatorname{Br} X$ is represented by an Azumaya algebra $\mathcal{A}_{\mathrm{Az}}$. Let n^2 denote the rank of $\mathcal{A}_{\mathrm{Az}}$ and $f \colon W \to X$ be the associated PGL_n-torsor. We complete the proof by noting that $\operatorname{ev}_{\mathcal{A}}^{-1}(0)$ is exactly equal to $f(W(k_v))$, which is open by the implicit function theorem. □

Proposition 3.4. *If X is smooth, then $X(\mathbb{A}_k)^{\mathrm{Br}}$ is closed in the adelic topology.*

Proof. Let $\mathcal{A} \in \operatorname{Br} X$ and let S be the finite set of primes for which the hypotheses of Lemma 3.1 are *not* satisfied. Let $(P_v) \in X(\mathbb{A}_k)$ be in the closure of $X(\mathbb{A}_k)^{\mathcal{A}}$. Thus, there exists a $(Q_v) \in X(\mathbb{A}_k)^{\mathcal{A}}$ such that

(1) $Q_v \in \mathcal{X}(\mathcal{O}_v)$ for all $v \notin S$ such that $P_v \in \mathcal{X}(\mathcal{O}_v)$, and
(2) Q_v is close enough to P_v so that $\operatorname{ev}_{\mathcal{A}}(P) = \operatorname{ev}_{\mathcal{A}}(P_v)$ for all $v \in S$ and all v such that $P_v \notin \mathcal{X}(\mathcal{O}_v)$.

Then by Lemma 3.1 $\operatorname{ev}_{\mathcal{A}}(P_v) = \operatorname{ev}_{\mathcal{A}}(Q_v)$ for all v so (P_v) is contained in $X(\mathbb{A}_k)^{\mathcal{A}}$. Thus $X(\mathbb{A}_k)^{\mathcal{A}}$ is closed and as $X(\mathbb{A}_k)^{\mathrm{Br}} = \cap_{\mathcal{A} \in \operatorname{Br} X} X(\mathbb{A}_k)^{\mathcal{A}}$, $X(\mathbb{A}_k)^{\mathrm{Br}}$ is also closed. □

3.2. **The étale-Brauer obstruction.** Let G be a finite étale group scheme over k and let $f \colon Y \to X$ be a fppf G-torsor over X.[1] Then, for any k-point $x \in X(k)$, the fiber Y_x is a G-torsor over k. As G-torsors over k are classified by $\mathrm{H}^1(k, G)$, we obtain a partition of $X(k)$ indexed by $\mathrm{H}^1(k, G)$, namely

$$X(k) = \coprod_{\tau \in \mathrm{H}^1(k,G)} \{x \in X(k) : [Y_x] = \tau\}. \tag{3.1}$$

For any cocycle τ representing a class in $\mathrm{H}^1(k, G)$, we may use contracted products to construct a $\overline{k}/k$-twist $f^\tau \colon Y^\tau \to X$ of $f \colon Y \to X$ such that

$$f^\tau(Y^\tau(k)) = \{x \in X(k) : [Y_x] = [\tau]\}.$$

(See [Sko01, §2.2] or [Poo14, §8.4] for details on the construction.) Therefore,

$$X(k) = \coprod_{[\tau] \in \mathrm{H}^1(k,G)} f^\tau(Y^\tau(k)). \tag{3.2}$$

Example 3.5. Let $Y \subset \mathbb{P}^5$ be the complete intersection cut out by the following equations:

$$\begin{aligned} s^2 &= xy + 5z^2, \\ s^2 - 5t^2 &= x^2 + 3xy + 2y^2, \\ u^2 &= 12x^2 + 111y^2 + 13z^2, \end{aligned}$$

and let $\sigma \colon \mathbb{P}^5 \to \mathbb{P}^5$ denote the involution $(s : t : u : x : y : z) \mapsto (-s : -t : -u : x : y : z)$. Note that since $V(s^2, s^2 - 5t^2, u^2) \subset \mathbb{P}^2$ and $V(xy + 5z^2, x^2 + 3xy + 2y^2, 12x^2 + 111y^2 + 13z^2)$

[1] If you are not familiar with torsors, you should think of Y as a scheme that "locally" over X is isomorphic to $G \times U$. See [Poo14, §6.5] or [Sko01, §2] for precise definitions.

are both empty, $\sigma|_Y$ has no fixed points. Thus the quotient map $f\colon Y \to X := Y/\sigma$ is a torsor under the finite étale group $\mathbb{Z}/2\mathbb{Z}$.

Let us consider $X(\mathbb{Q})$. A $\mathbb{Q}$-point on X is the image of a degree 2 zero-dimensional closed subscheme on Y whose $\overline{\mathbb{Q}}$-points are interchanged by σ, i.e.,

$$X(\mathbb{Q}) = \coprod_{[d]\in\mathbb{Q}^\times/\mathbb{Q}^{\times 2}} \left\{ \{P,\sigma(P)\} : P \in Y(\mathbb{Q}(\sqrt{d})), \tau(P) \in \{P,\sigma(P)\}\ \forall\tau \in \operatorname{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \right\}.$$

The condition that $\tau(P) \in \{P,\sigma(P)\}$ for all $\tau \in \operatorname{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ implies that $P = (s_0\sqrt{d} : t_0\sqrt{d} : u_0\sqrt{d} : x_0 : y_0 : z_0)$ where $(s_0 : t_0 : u_0 : x_0 : y_0 : z_0)$ is a $\mathbb{Q}$-point on a variety Y^d cut out by the equations

$$\begin{aligned} ds^2 &= xy + 5z^2, \\ d(s^2 - 5t^2) &= x^2 + 3xy + 2y^2, \\ du^2 &= 12x^2 + 111y^2 + 13z^2. \end{aligned} \tag{3.3}$$

In other words,

$$X(\mathbb{Q}) = \coprod_{[d]\in\mathbb{Q}^\times/\mathbb{Q}^{\times 2}} f^d(Y^d(\mathbb{Q})),$$

where $f^d\colon Y^d \to X$ is obtained by quotienting by $\sigma|_{Y^d}$. ⋄

The partition given in (3.2) is preferable to that in (3.1), despite the additional machinery required, since (3.2) allows us to use Brauer-Manin conditions on Y^τ to obtain conditions on X. Indeed, (3.2) implies that

$$X(k) \subset \bigcup_{[\tau]\in \mathrm{H}^1(k,G)} f^\tau(Y^\tau(\mathbb{A}_k)^{\mathrm{Br}}).$$

Furthermore, this containment holds for *any* torsor $f\colon Y \to X$ under a finite étale group G. Thus, we have

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{et,Br}} := \bigcap_{\substack{f\colon Y\to X \text{ torsor} \\ \text{under finite étale } G}} \bigcup_{[\tau]\in \mathrm{H}^1(k,G)} f^\tau(Y^\tau(\mathbb{A}_k)^{\mathrm{Br}}). \tag{3.4}$$

The set $X(\mathbb{A}_k)^{\mathrm{et,Br}}$ is called the étale-Brauer set of X. It can be strictly smaller than the Brauer-Manin set [Sko99], so gives a stronger obstruction.

Proposition 3.6. *Assume that X is proper. For any torsor $f\colon Y \to X$ under a finite étale k-group G, $Y^\tau(\mathbb{A}_k) = \emptyset$ for all but finitely many $[\tau] \in \mathrm{H}^1(k,G)$.*

Proof. This is [Sko01, Prop. 5.3.2]. □

Corollary 3.7. *If X is proper, then $X(\mathbb{A}_k)^{\mathrm{et,Br}}$ is closed in the adelic topology.*

Proof. This is [Sko01, Prop. 5.3.3]. □

3.3. **Beyond étale-Brauer.** In 2010, Poonen constructed a Châtelet surface bundle X over a curve C/k such that $X(\mathbb{A}_k)^{\text{et,Br}} \neq \emptyset$ and $X(k) = \emptyset$. This example demonstrates that the étale-Brauer obstruction is *insufficient* to explain all counterexamples to the Hasse principle. Similar examples have since been constructed by Harpaz–Skorobogatov [HS14], Colliot-Thélène–Pál–Skorobogatov [CTPS], and Smeets [Sme]; each successive example shows that the étale-Brauer obstruction is *still* insufficient to explain all failures of the Hasse principle even if we restrict to a smaller class of projective varieties $\mathcal{S}$. For example, it is not enough to restrict to:

- varieties of dimension at most 2 [HS14],
- varieties which are uniruled [CTPS],
- varieties with trivial Albanese variety [Sme], or,
- if the *abc* conjecture holds, simply connected varieties [Sme].

The above examples have a fundamental feature in common: X is obtained as a fiber product $Y \times_{\mathbb{P}^1} Z$ with the following properties

(1) the image of $Y(k)$ in $\mathbb{P}^1(k)$ is a finite set T and
(2) for all $t \in T$, the fiber of $Z \to \mathbb{P}^1$ above t has no rational points.

These properties immediately imply that $X(k) = \emptyset$. Indeed, by factoring the map $X \to \mathbb{P}^1$ through Y, property (1) implies that all k-points of X lie over T; however, by factoring the map $X \to \mathbb{P}^1$ through Z, property (2) shows that no k-points lie over T.

The difficulty in the constructions is in ensuring that X has the desired properties (e.g., 2-dimensional, uniruled, trivial Albanese, etc.) and that $X(\mathbb{A}_k)^{\text{et,Br}} \neq \emptyset$, and in this step the arguments can vary significantly. See [Poo14, §8.6.2] for a nice explanation (with pictures!) of a slight variation on the example of Colliot-Thélène, Pál, and Skorobogatov [CTPS, §3].

4. Geometry and arithmetic

We are interested in determining classes of varieties $\mathcal{S}$, defined by some geometric property, for which

(1) $\mathcal{S}$ satisfies the Hasse principle, i.e., $X(\mathbb{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset \; \forall X \in \mathcal{S}$,
(2) the Brauer-Manin obstruction is the only obstruction to the Hasse principle for $\mathcal{S}$, i.e., $X(\mathbb{A}_k)^{\text{Br}} \neq \emptyset \Rightarrow X(k) \neq \emptyset \; \forall X \in \mathcal{S}$, or
(3) the étale-Brauer obstruction is the only obstruction to the Hasse principle for $\mathcal{S}$, i.e., $X(\mathbb{A}_k)^{\text{et,Br}} \neq \emptyset \Rightarrow X(k) \neq \emptyset \; \forall X \in \mathcal{S}$.

Remark 4.1. We will sometimes say "the Brauer-Manin obstruction (or the étale-Brauer obstruction) is sufficient to explain all failures of the Hasse principle for $\mathcal{S}$" instead of "the Brauer-Manin obstruction (or the étale-Brauer obstruction) is the only obstruction to the Hasse principle for $\mathcal{S}$".

Example 4.2. The Hasse-Minkowski theorem says that all quadric hypersurfaces satisfy the Hasse principle. ⋄

Example 4.3. If the Tate-Shafarevich group of any elliptic curve is finite, then the Brauer-Manin obstruction is sufficient to explain all failures of the Hasse principle on genus 1 curves [Man71]. More precisely, if C is a genus 1 curve over k, and the Tate–Shafarevich group of the Jacobian of *this particular curve* C is finite, then $C(k) \neq \emptyset$ if and only if $C(\mathbb{A}_k)^{\text{Br}} \neq \emptyset$. ⋄

By the Lang-Nishimura lemma [Lan54, Nis55] (also see [VA13, Lemma 1.1]), the existence of a smooth k-rational point is a (k-)birational invariant of proper integral varieties. Thus, it will be helpful to understand the classification of smooth proper surfaces up to birational equivalence. As every smooth proper surface is projective [Zar58b], we reduce to considering smooth projective surfaces.

Remark 4.4. This section was heavily influenced by Beauville's book *Complex Algebraic Surfaces* [Bea96], Hassett's article *Rational surfaces over nonclosed fields* [Has09], and Várilly-Alvarado's article *Arithmetic of del Pezzo surfaces* [VA13].

4.1. **Kodaira dimension and other numerical invariants.** Let X be a smooth projective variety and let ω_X denote the canonical sheaf. The Kodaira dimension of X, denoted $\kappa(X)$ is $\max_n \dim(\phi_{\omega_X^{\otimes n}}(X))$ if $\mathrm{H}^0(X, \omega_X^{\otimes n}) \neq \emptyset$ for some n, and $-\infty$ otherwise. Note that $\kappa(X) \in \{-\infty, 0, \dots, \dim(X)\}$.

Exercise 4.5. Let X be a curve. Prove that

$$\kappa(X) = -\infty \Leftrightarrow g(X) = 0, \quad \kappa(X) = 0 \Leftrightarrow g(X) = 1, \quad \text{and} \quad \kappa(X) = 1 \Leftrightarrow g(X) \geq 2.$$

◇

The Kodaira dimension of X gives a rough measure of the geometric complexity of X. Varieties of smaller Kodaira dimension are typically easier to classify. Varieties X of maximal Kodaira dimension, i.e., with $\dim(X) = \kappa(X)$, are said to be of general type.

To state the classification of surfaces of low Kodaira dimension, we will need some additional numerical invariants. The irregularity of X is $q(X) := \dim \mathrm{Alb}_X$, where Alb_X denotes the Albanese variety of X, and the geometric genus of X is $p_g(X) := \mathrm{h}^0(X, \omega_X)$. More generally, we may consider the plurigenera of X $\{P_n(X) := \mathrm{h}^0(X, \omega_X^{\otimes n})\}_{n \geq 1}$.

The Betti numbers of X are $b_i := \dim \mathrm{H}^i(X^{\mathrm{sep}}, \mathbb{Q}_\ell)$, $i = 0, \dots 4$, where ℓ is a prime different from the characteristic of k; they are independent of the choice of ℓ. We write $\mathrm{Pic}\, X$ for the Picard group of X and $\mathrm{NS}\, X$ for the Néron-Severi group of X, i.e., the group of divisors modulo algebraic equivalence. We write $\rho(X)$ for the (geometric) Picard rank of X, i.e., $\rho(X) := \mathrm{rank}\, \mathrm{NS}\, \mathrm{Pic}\, X^{\mathrm{alg}}$.

4.2. **Geometry of surfaces of Kodaira dimension $-\infty$.**

Theorem 4.6 (Enriques [Enr49] in characteristic 0, Mumford [Mum69] in characteristic p)**.** *Let X be a smooth projective surface over a field k. If $\kappa(X) = -\infty$ then $X_{k^{\mathrm{alg}}}$ is ruled, i.e., there exist a curve C and a morphism (over k^{alg}) $f\colon X_{k^{\mathrm{alg}}} \to C$ whose generic fiber is isomorphic to $\mathbb{P}^1_{\mathbf{k}(C)}$.*

Since there are no rational curves on abelian varieties, any map $\overline{X} \to A$, where A is an abelian variety, factors through C. Thus, by the universal property of the Albanese variety, $\mathrm{Alb}(X)$ must be isomorphic to $\mathrm{Jac}(C)$. In particular, if C has positive genus, then the image of $X \to \mathrm{Alb}(X)$ is a curve that is geometrically isomorphic to C and the generic fiber of $X \to \mathrm{im}(X \to \mathrm{Alb}(X))$ is a curve of genus 0. As the map $X \to \mathrm{Alb}(X)$ is defined over k, we have the following.

Proposition 4.7. *If $X_{k^{\mathrm{alg}}}$ is ruled over a curve of positive genus, then there exists a higher genus curve C/k and a morphism $f\colon X \to C$, defined over k, whose generic fiber is a genus 0 curve, i.e., X is a conic bundle over a positive genus curve.*

By [Bea96, Prop. III.21], if $\kappa(X) = -\infty$ then $X_{k^{\mathrm{alg}}}$ is ruled over a positive genus curve if and only if $q(X) > 0$. Let us now consider smooth projective surfaces X with $\kappa(X) = -\infty$ and $q(X) = 0$.

Theorem 4.8 (Castelnuovo, Zariski [Zar58a] in positive characteristic, see also [Lan81])**.** *Let X be a smooth projective surface. Then $P_2(X) = 0$ and $q(X) = 0$ if and only if $X_{k^{\mathrm{alg}}}$ is birational to $\mathbb{P}^2$.*

Remark 4.9. Castelnuovo and Zariski proved the forwards direction. The backwards direction follows since P_2 and and q are birational invariants.

A theorem of Iskovskikh further partitions such surfaces.

Theorem 4.10 ([Isk79, Thm. 1])**.** *Let X be a smooth projective geometrically rational surface over k. Then ω_X^{-1} is ample or X is k-birational to a conic bundle over a genus 0 curve.*

Remark 4.11. The two possibilities are not exclusive. For example, $\mathbb{P}^1 \times \mathbb{P}^1$ has ample anti-canonical sheaf and can be realized as a conic bundle over $\mathbb{P}^1$.

We say that a smooth projective surface X is a del Pezzo surface if ω_X^{-1} is ample. The degree of a del Pezzo surface is the self-intersection number $\omega_X \cdot \omega_X$.

Theorem 4.12 ([Man86, Thm. 24.4] if k is perfect, [VA13, Thm. 1.6] in general)**.** *Let X be a del Pezzo surface of degree d over k. Then $X_{k^{\mathrm{sep}}}$ is isomorphic to the blow-up of $\mathbb{P}^2_{k^{\mathrm{sep}}}$ at $9-d$ points in general position or $d = 8$ and $X_{k^{\mathrm{sep}}}$ is isomorphic to $\mathbb{P}^1_{k^{\mathrm{sep}}} \times \mathbb{P}^1_{k^{\mathrm{sep}}}$. In particular, $1 \leq d \leq 9$.*

If X is a del Pezzo surface of degree $d \geq 3$, then the anticanonical sheaf is very ample and embeds X as a smooth degree d surface in $\mathbb{P}^d$. If X is a degree 2 del Pezzo surface, then the anticanonical map $f\colon X \to \mathbb{P}^2$ realizes X as a smooth double cover of $\mathbb{P}^2$ ramified along a quartic curve. If X is a degree 1 del Pezzo surface, then X can be realized as a smooth sextic hypersurface in the weighted projective space $\mathbb{P}(1,1,2,3)$; see [VA13, §1.5.4] for more details.

4.3. **Arithmetic of surfaces of Kodaira dimension $-\infty$.** As discussed in the previous section, a surface of Kodaira dimension $-\infty$ is birational to one of

(1) A conic bundle over a curve of genus ≥ 1,
(2) A conic bundle over a curve of genus 0, or
(3) a del Pezzo surface of degree $1 \leq d \leq 9$.

As explained in §3.3, Colliot-Thélène, Pál and Skorobogatov [CTPS] have demonstrated that the étale-Brauer obstruction is *insufficient* to explain all failures of the Hasse principle for (1). This is in stark contrast to our expectations for (2) and (3). Indeed, we have

Conjecture 4.13 ([CTS80])**.** *The (algebraic) Brauer-Manin obstruction is the only obstruction to the Hasse principle (and weak approximation) for geometrically rational surfaces.*

Remark 4.14. The Brauer group of a geometrically rational surface is purely algebraic, so the Brauer-Manin set and the algebraic Brauer-Manin set are equal.

Classical results imply that del Pezzo surfaces of degree at least 5 satisfy the Hasse principle (see e.g. [VA13, §2] for proofs) so Conjecture 4.13 holds in these cases.[2] In fact, del Pezzo surfaces of degree 5 and 7 are known to always have a rational point! We note that del Pezzo surfaces of degree at least 5 have trivial Brauer group [Man86, Thm. 29.3] so satisfying the Hasse principle is (in this case) equivalent to sufficiency of the Brauer-Manin obstruction.

Del Pezzo surfaces of degree at most 4 can have non-trivial Brauer group and for $d = 4, 3$ or 2, there exist degree d del Pezzo surfaces that fail the Hasse principle [BSD75, SD62, KT04]. Degree 1 del Pezzo surfaces always have a k-rational point since $\omega_X^{\otimes -1}$ has a unique base point; however, degree 1 del Pezzo surfaces can fail to satisfy weak approximation [VA08].

Conjecture 4.13 is known for very few cases of del Pezzos surfaces of degree at most 4, even conditionally. The strongest result in this direction is due to Wittenberg [Wit07], building on ideas of Swinnerton-Dyer [SD95]. Wittenberg proves that "sufficiently general" del Pezzo surfaces of degree 4 satisfy Conjecture 4.13 *assuming* Schinzel's hypothesis and finiteness of Tate-Shafarevich groups of elliptic curves over number fields [Wit07, Thm. 3.2].

In the case of rational conic bundles, Conjecture 4.13 if the conic fibration has at most 5 degenerate fibers [CTSSD87a, CTSSD87b, CT90, SS91] or if the degenerate fibers of the fibration lie over $\mathbb{Q}$-rational points [BMS14]. This latter result of Browning, Matthiesen, and Skorobogatov was the first result for rational conic bundles with *no* assumption on the number of degenerate fibers.

4.4. **Surfaces of Kodaira dimension** 0. In this section, we restrict to the case that k has characteristic 0. For the results in positive characteristic, see [Mum69, BM77, BM76].

Theorem 4.15 ([Bea96, Thm. VIII.2])**.** *Let k be a field of characteristic 0 and let X/k be a minimal surface of Kodaira dimension 0. Then X lies in exactly one of the following cases.*

1. $p_g = q = 0$. *In this case $\omega_X^{\otimes 2} \cong \mathcal{O}_X$ and X is an* Enriques surface*.*
2. $p_g = 1, q = 0$. *In this case $\omega_X \cong \mathcal{O}_X$ and X is a* K3 surface*.*
3. $p_g = 0, q = 1$. *In this case X is a* bielliptic surface*, i.e., $X_{k^{\mathrm{alg}}} \cong (E_1 \times E_2)/G$ where G is a finite group of translations of E_1 such that $E_2/G \cong \mathbb{P}^1$.*
4. $p_g = 1, q = 2$. *In this case X is a twist of an Abelian surface.*

Example 4.16. Smooth quartic surfaces in $\mathbb{P}^3$, smooth complete intersections of a quadric and a cubic in $\mathbb{P}^4$, and smooth complete intersections of 3 quadrics in $\mathbb{P}^5$ are all examples of K3 surfaces.

Assuming finiteness of Tate-Shafarevich groups, we completely understand failures of the Hasse principle on bielliptic surfaces and twists of abelian surfaces.

Theorem 4.17. [Lin40, Rei42, Man71, Sko99] *Assume that Tate-Shafarevich groups of abelian surfaces are finite. Then*

1. *twists of abelian surfaces may fail the Hasse principle,*
2. *the Brauer-Manin obstruction is sufficient to explain all failures of the Hasse principle on twists of abelian surfaces,*
3. *the Brauer-Manin obstruction is* insufficient *to explain all failures of the Hasse principle on bielliptic surfaces, and*

[2] Del Pezzo surfaces of degree at least 5 are k-birational to $\mathbb{P}^2$ as soon as they have a rational point, so del Pezzos surfaces of degree at least 5 satisfy weak approximation as soon as they satisfy the Hasse principle.

(4) *the étale-Brauer obstruction is sufficient to explain all failures of the Hasse principle on bielliptic surfaces.*

Proof. For (1), we may take $X = C \times C$ where C is the counterexample of Lind and Reichardt from §1. Manin proved (2)[Man71] and Skorobogatov constructed a bielliptic surface that demonstrates (3) [Sko99]. It remains to prove (4).

Let X be a bielliptic surface. Then ω_X is nontrivial and $\omega_X^{\otimes 4}$ or $\omega_X^{\otimes 6}$ is trivial [Bea96, Cor. VIII.7]. Therefore, ω_X defines a nontrivial finite étale cover $\pi\colon X' \to X$. Since ω'_X is trivial and $\chi(\mathcal{O}_{X'}) = \deg(\pi)\chi(\mathcal{O}_X)$[Bea96, Lemma VI.3], X' is a twist of an Abelian surface. By (3.2), $X(k) \neq \emptyset$ if and only if $X'^\tau(k) \neq \emptyset$ for some τ. Since X'^τ is a twist of an abelian surface, Manin's result implies that $X(k) \neq \emptyset$ if and only if $X'^\tau(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ for some τ. This completes the proof. □

The above proof shows that the study of bielliptic surfaces is naturally related to the study of (twists of) abelian surfaces. A similar statement holds for Enriques surfaces and K3 surfaces.

Theorem 4.18. [Bea96, Prop. III.17] *Let Y be a K3 surface and let $\sigma\colon Y \to Y$ be a fixed point free involution. Then $X = Y/\sigma$ is an Enriques surface. Conversely, if X is an Enriques surface, then there is a K3 surface Y and an étale degree 2 morphism $f\colon Y \to X$.*

K3 surfaces are simply connected so have no nontrivial étale covers. Thus, the Brauer-Manin obstruction on a K3 surface is the same as the étale-Brauer obstruction. A similar argument to that in Theorem 4.17 yields:

Corollary 4.19. *If the Brauer-Manin obstruction is sufficient to explain all failures of the Hasse principle for K3 surfaces, then the étale-Brauer obstruction is sufficient to explain all failures of the Hasse principle for Enriques surfaces.*

Skorobogatov has conjectured that the Brauer-Manin obstruction is indeed sufficient to explain all failres of the Hasse principle and weak approximation for K3 surfaces [Sko09].

Remark 4.20. The Brauer-Manin obstruction is necessary to explain all failures of the Hasse principle and weak approximation for K3 surfaces. Indeed, even the algebraic Brauer-Manin obstruction does not suffice[Wit04, HVA13]. We also know that the étale-Brauer obstruction is necessary to explain all failures of the Hasse principle for Enriques surfaces [VAV11, BBM+].

5. A detailed example

The goal of this section is to sketch the proof of the following theorem.

Theorem 5.1 ([VAV11, BBM+])**.** *Let $Y \subset \mathbb{P}^5$ be the K3 surface cut out by the following equations*

$$s^2 = xy + 5z^2,$$
$$s^2 - 5t^2 = x^2 + 3xy + 2y^2,$$
$$u^2 = 12x^2 + 111y^2 + 13z^2,$$

let $\sigma\colon \mathbb{P}^5 \to \mathbb{P}^5$ denote the involution $(s:t:u:x:y:z) \mapsto (-s:-t:-u:x:y:z)$, and let X denote the Enriques surface Y/σ. Then

$$X(\mathbb{A}_\mathbb{Q})^{\mathrm{et,Br}} = \emptyset \quad \text{and } X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \neq \emptyset.$$

In §5.1, we prove that $Y(\mathbb{A}_\mathbb{Q})$ is nonempty and in §5.2, we prove that $X(\mathbb{A}_\mathbb{Q})^{\mathrm{et},\mathrm{Br}} = \emptyset$. Next in §5.3, we prove that $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}_1} \neq \emptyset$ and finally in §5.4, we prove that $\mathrm{Br}\, X = \mathrm{Br}_1 X$ and so $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \neq \emptyset$.

5.1. **Local solubility.**

Proposition 5.2. *We have $Y(\mathbb{A}_\mathbb{Q}) \neq \emptyset$, so $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$.*

Proof. Let us first recall the statement of the Weil conjectures. For any smooth projective n-dimensional variety V over $\mathbb{F}_q$, we may define

$$\zeta(V, s) = \exp\left(\sum_{m=0}^{\infty} \frac{\#V(\mathbb{F}_{q^m})}{m} q^{-sm}\right).$$

Then the Weil conjectures (now theorems [Del74]) state that:

(1) $\zeta(V, s) = \prod_{i=0}^{2n} P_i(q^{-s})^{(-1)^{i+1}}$ where $P_i(T) \in \mathbb{Z}[T]$, and over $\mathbb{C}$, $P_i(T)$ factors as $\prod_j(1-\alpha_{i,j}T)$ with $P_0(T) = 1 - T$ and $P_{2n} = 1 - q^n T$.
(2) $\zeta(V, n-s) = \pm q^{\chi(V)(\frac{n}{2}-s)}\zeta(V, s)$,
(3) $|\alpha_{i,j}| = q^{i/2}$ for all $i \in [0, 2n]$ and all j, and
(4) If V is the reduction of a smooth variety W over a global field, then $\deg(P_i) = b_i(W(\mathbb{C}))$ where b_i denotes the i^{th} Betti number.

Exercise 5.3. Show that (1) implies that

$$\#V(\mathbb{F}_{q^m}) = \sum_{i,j}(-1)^i \alpha_{i,j}^m,$$

for all $m \geq 0$. ⋄

By applying the Jacobian criterion over $\mathbb{Z}$, we see that $Y/\mathbb{Q}$ is smooth and that Y has good reduction modulo p for all p outside of

$$\{2, 3, 5, 13, 37, 59, 151, 157, 179, 821, 881, 1433\}.$$

Exercise 5.4. Verify the above claim. (You may want to use Magma or Macaulay2.) ⋄

Let p be a prime of good reduction for Y. The Betti numbers of any K3 surface are $b_0 = 1, b_1 = 0, b_2 = 22, b_3 = 0, b_4 = 1$. Thus, the Weil conjectures imply that

$$\#Y(\mathbb{F}_p) = 1 + p^2 + \sum_{j=1}^{22} \alpha_{2,j}.$$

Since $|\alpha_{2,j}| = p$, we must have that $|\#Y(\mathbb{F}_p) - 1 - p^2| \leq 22p$. In particular, if $p \geq 23$ then $\#Y(\mathbb{F}_p) \neq \emptyset$. Since Y has good reduction at $\mathbb{F}_p$, any $\mathbb{F}_p$-point lifts to a $\mathbb{Q}_p$ point by Hensel's Lemma [Poo14, Thm. 3.5.54]. Thus, $Y(\mathbb{Q}_p) \neq \emptyset$ for all p outside of

$$\{2, 3, 5, 7, 11, 13, 17, 19, 37, 59, 151, 157, 179, 821, 881, 1433\}.$$

One can check that $Y(\mathbb{R}) \neq \emptyset$ and that for $p > 5$ in our remaining list Y has a smooth $\mathbb{F}_p$-point. Then we can use Hensel's Lemma to verify that we have the following points over

$\mathbb{Q}_2, \mathbb{Q}_3$, and $\mathbb{Q}_5$:

$$\left(\sqrt{129} : 2\sqrt{21/5} : \sqrt{2113} : 1 : 4 : 5\right) \in Y(\mathbb{Q}_2),$$
$$\left(0 : 0 : \sqrt{821/5} : -2 : 1 : \sqrt{2/5}\right) \in Y(\mathbb{Q}_3),$$
$$\left(1 : 2\sqrt{-1} : \sqrt{1801} : 1 : -4 : 1\right) \in Y(\mathbb{Q}_5).$$

Thus $Y(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$. □

5.2. **Lack of $\mathbb{Q}$-points.**

Proposition 5.5. $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{et,Br}} = \emptyset$.

Proof. Recall that

$$X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{et,Br}} = \left(\bigcup_{[d]\in\mathbb{Q}^\times/\mathbb{Q}^{\times 2}} f^d(Y^d(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}) \right) \cap X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}$$

and that Y^d is as in (3.3).

Exercise 5.6. Prove that if d is a squarefree integer divisible by a prime p different from 2 and 5, then $Y^d(\mathbb{Q}_p) = \emptyset$. ◇

Exercise 5.7. Prove that if $d < 0$, then $Y^d(\mathbb{R}) = \emptyset$. ◇

Exercise 5.8. Prove that if $d \equiv 2 \bmod 3$, then $Y^d(\mathbb{Q}_3) = \emptyset$. ◇

Exercise 5.9. Prove that $Y^{10}(\mathbb{Q}_5) = \emptyset$. ◇

From the above exercises, we see that $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{et,Br}} = f(Y(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}})$. Thus it remains to prove that $Y(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$. Note that Y is a double cover of the degree 4 del Pezzo surface $S \subset \mathbb{P}^4$ given by

$$s^2 = xy + 5z^2,$$
$$s^2 - 5t^2 = x^2 + 3xy + 2y^2,$$

which was studied by Birch and Swinnerton-Dyer [BSD75]. Let g denote the double cover morphism $Y \to S$. The functoriality of the Brauer group shows that $g(Y(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}) \subset S(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}$. Thus, the following lemma completes the proof of Proposition 5.5. □

Lemma 5.10 ([BSD75, §§4,6])**.** *The algebra $\mathcal{A} := (5, (x+y)/x)_2 \in \operatorname{Br} \mathbf{k}(S)$ is in the image of $\operatorname{Br} S$ and $S(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} = \emptyset$.*

Proof. Since $xy = s^2 - 5z^2$ and $(x+y)(x+2y) = s^2 - 5t^2$ on S, by (2.2) we have

$$\mathcal{A} = \left(5, \frac{x+y}{x}\right)_2 = \left(5, \frac{x+2y}{x}\right)_2 = \left(5, \frac{x+y}{y}\right)_2 = \left(5, \frac{x+2y}{y}\right)_2 \in \operatorname{Br} \mathbf{k}(S).$$

Thus, by the exact sequence (2.3) and Lemma 2.10, $\mathcal{A} \in \operatorname{Br} S$. Let us compute $S(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}}$. If v is split in $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$ (including $v = \infty$), then $5 \in \mathbb{Q}_v^{\times 2}$ and so $\operatorname{inv}_v(\mathcal{A}(P_v)) = 0$ for all $P_v \in S(\mathbb{Q}_v)$.

Let v be a finite place that is inert in $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$. Let $P = (s_0 : t_0 : x_0 : y_0 : z_0) \in S(\mathbb{Q}_v)$ and assume that $y_0 \neq 0$. By the strong triangle inequality

$$0 = v(1) = v\left(\frac{x_0 + 2y_0}{y_0} - \frac{x_0 + y_0}{y_0}\right) \geq \min\left\{v(x_0/y_0 + 2), v(x_0/y_0 + 1)\right\}.$$

Thus, if $v(x_0) \geq v(y_0)$ then at least one of $v(x_0/y_0 + 2), v(x_0/y_0 + 1)$ is equal to 0. Thus $\mathrm{inv}_v(\mathcal{A}(P_v)) = 0$ by Example 2.7.

Now let $P_v = (s_0 : t_0 : x_0 : y_0 : z_0) \in S(\mathbb{Q}_v)$ with $v(x_0) < v(y_0)$. Then a similar argument shows that one of $\{1 + y_0/x_0, 1 + 2y_0/x_0\}$ is a v-adic unit and so $\mathrm{inv}_v(\mathcal{A}(P_v)) = 0$. Since there are no $\mathbb{Q}_v$-points of S with $x_0 = y_0 = 0$, this shows that $\mathrm{inv}_v(\mathcal{A}(P_v)) = 0$ for all $P_v \in S(\mathbb{Q}_v)$ and all inert v.

It remains to consider the case when $v = 5$. Let $P = (s_0 : t_0 : x_0 : y_0 : z_0) \in S(\mathbb{Q}_5)$. Without loss of generality we may assume that $v(s_0), v(t_0), v(x_0), v(y_0), v(z_0)$ are all non-negative and that at least one is 0. Then modulo 5 we have that

$$s_0^2 \equiv x_0 y_0 \equiv x_0^2 + 3x_0 y_0 + 2y_0^2.$$

In particular, if one of x_0 or y_0 is 0 modulo 5, then x_0, y_0 and s_0 are 0 modulo 5. However, if $x_0 \equiv y_0 \equiv s_0 \bmod 5$, then $5z_0^2 \equiv -5t_0^2 \equiv 0 \bmod 25$ which results in a contradition. Thus, we have $x_0, y_0 \neq 0 \bmod 5$.

Since $x_0^2 + 2x_0 y_0 + 2y_0^2 \equiv 0 \bmod 5$, x_0/y_0 is equivalent to 1 or 2 modulo 5. Further, since $(s_0/y_0)^2 \equiv x_0/y_0 \bmod 5$, x_0/y_0 is a square modulo 5 and hence must be 1 modulo 5. Then $(x_0 + y_0/y_0) \equiv 2 \bmod 5$ and so $(x_0 + y_0)/y_0$ is not a norm from $\mathbb{Q}_5(\sqrt{5})$. Hence, we have

$$\mathrm{inv}_v(\mathcal{A}(P_v)) = \begin{cases} 0 & \text{if } v \neq 5 \\ \frac{1}{2} & \text{if } v = 5 \end{cases}$$

and so $S(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} = \emptyset$. □

5.3. **Nonempty algebraic Brauer set.** The goal of this section is to prove that $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} \neq \emptyset$. To do so we must first determine $\mathrm{Br}_1 X / \mathrm{Br}\,\mathbb{Q}$. Recall (see (2.4)) that we have an isomorphism

$$\frac{\mathrm{Br}_1 X}{\mathrm{Br}_0 X} \xrightarrow{\sim} \mathrm{H}^1\left(G_{\mathbb{Q}}, \mathrm{Pic}\,\overline{X}\right).$$

The (geometric) Picard group of an Enriques surface is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}^{10}$ and the quotient $\mathrm{Num}\,\overline{X}$ with its intersection pairing is isomorphic to $U \oplus E_8(-1)$, the unique even unimodular lattice of rank 10 and signature $(1, 9)$[CD89, Thms. 1.2.1, 2.5.1 and Prop. 1.2.1]. To compute $\mathrm{Pic}\,\overline{X}$ as a Galois module, we must find explicit curves which generate the Picard group.

Consider the degeneracy locus of the net of quadrics corresponding to Y, i.e., consider

$$Z : \det(\lambda M_1 + \mu M_2 + \nu M_3) = 0,$$

where M_1, M_2, and M_3 are the symmetric matrices associated to a basis Q_1, Q_2, Q_3 of the net of quadrics. Since each matrix M_i is a block matrix, with four blocks of size 1 (corresponding to s^2, t^2, u^2, and z^2) and one block of size 2, one can easily deduce that Z is a union of 4 lines and a conic. The singular points of Z correspond to rank 4 quadrics in the net. For example, the 6 singular points obtained from an intersection of 2 lines correspond to the rank 4 quadrics

$$\begin{array}{cc} s^2 - xy - 5z^2, & s^2 - 5t^2 - (x+y)(x+2y), \\ u^2 - 12x^2 - 111y^2 - 13z^2, & 13s^2 - 5u^2 + 60x^2 - 13xy + 555y^2, \\ 5t^2 + x^2 + 2xy + 2y^2 - 5z^2, & 65t^2 - 5u^2 + 73x^2 + 26xy + 581y^2. \end{array}$$

Each rank 4 quadric will give us two pencils of genus 1 curves on Y. Indeed, over $\overline{\mathbb{Q}}$ any rank 4 quadric may be written as $\ell_0\ell_1 - \ell_2\ell_3$ for some linear forms ℓ_i. Hence any rank 4 quadric in 6 variables yields (over $\overline{\mathbb{Q}}$) two pencils of 3-planes: $V(\ell_0 - T\ell_2, \ell_1 - \frac{1}{T}\ell_3)$ and $V(\ell_0 - T\ell_3, \ell_1 - \frac{1}{T}\ell_2)$. Intersecting with Y yields two pencils of genus 1 curves, each fiber realized as the intersection of two quadrics with a 3-plane. Thus we obtain 28 classes of curves on Y which we denote by $F_1, G_1, \ldots, F_{14}, G_{14}$, where for each i, F_i and G_i are obtained from the same rank 4 quadric.

Exercise 5.11. Prove that for all i, $F_i + G_i$ is equal to the hyperplane section and that $\{F_i, G_i\}$ is fixed by σ. Verify that

$$\#\{i : \sigma(F_i) = F_i, \sigma(G_i) = G_i\} = 9.$$

Possibly after relabeling, we will assume that $F_1, G_1, \ldots F_9, G_9$ are fixed by σ.

The curve classes fixed by σ are exactly the classes that descend to X. Indeed, one can check that for every F_i there exists a morphism $\phi_i \colon X \to \mathbb{P}^1$ such that the following diagram commutes

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle |F_i|} & & \downarrow{\scriptstyle \phi_i} \\ \mathbb{P}^1 & \xrightarrow{T \mapsto T^2} & \mathbb{P}^1 \end{array}$$

Each ϕ_i has exactly two double fibers; all other fibers are reduced. Let C_i and $\widetilde{C}_i$ denote the classes of the reduced parts of the nonreduced fibers of ϕ_i. Additionally, for all G_i there exists a map $\psi_i \colon X \to \mathbb{P}^1$ with the analogous properties; we write D_i and $\widetilde{D}_i$ for the classes of the reduced parts of the nonreduced fibers of ψ_i. After possibly interchanging D_i and $\widetilde{D}_i$, we may assume that

$$C_i + D_i = \widetilde{C}_j + \widetilde{D}_j,\ 2(C_i - \widetilde{C}_i) = 2(D_i - \widetilde{D}_i) = 0,\ f^*C_i = f^*\widetilde{C}_i = F_i, \text{ and } f^*D_i = f^*\widetilde{D}_i = G_i$$

for all i and j.

Exercise 5.12. Prove that for all $i \neq j$

$$F_i^2 = G_i^2 = 0,\ F_i \cdot G_j = F_i \cdot F_j = G_i \cdot G_j = 2, \text{ and } F_i \cdot G_i = 4.$$

Then use the projection formula [Liu02, p. 399] to deduce that for all $i \neq j$

$$C_i^2 = D_i^2 = 0,\ C_i \cdot D_j = C_i \cdot C_j = D_i \cdot D_j = 1, \text{ and } C_i \cdot D_i = 2.$$

◇

These intersection numbers show that the sublattice of $\operatorname{Pic} \overline{Y}$ generated by $G_1, F_1, \ldots, F_{14}$ has rank 15 and discriminant 2^{17}. Further, the sublattice of $\operatorname{Num} \overline{X}$ generated by $D_1, C_1, \ldots, C_9$ has rank 10 and discriminant 4. However, $\operatorname{Num} \overline{X}$ is an even unimodular lattice of rank 10, so $\langle D_1, C_1, \ldots, C_9\rangle$ is a proper lattice. A computation shows that there are exactly two even unimodular lattices in $\operatorname{Num} \overline{X} \otimes \mathbb{Q}$ that contain $\langle D_1, C_1, \ldots, C_9\rangle$, namely

$$L_1 := \left\langle \frac{(D_1 + C_2 + \cdots + C_9)}{2}, C_1, \ldots, C_9 \right\rangle, \text{ and } L_2 := \left\langle \frac{(C_1 + C_2 + \cdots + C_9)}{2}, D_1, \ldots, C_9 \right\rangle.$$

The action of the absolute Galois group $G_{\mathbb{Q}}$ on these curve classes is described in [VAV11, Table A.2]. We note that the action factors through a 2-group, so $\mathrm{H}^1(G_{\mathbb{Q}}, L_i)$ is a 2 group. Equipped with this table, we can compute that for $i = 1, 2$

$$\mathrm{H}^1(G_{\mathbb{Q}}, L_i)[2] \cong \frac{(L_i/2L_i)^{G_{\mathbb{Q}}}}{L_i^{G_{\mathbb{Q}}}/2L_i^{G_{\mathbb{Q}}}} = 0.$$

Thus, regardless of whether $\operatorname{Num} \overline{X} = L_1$ or L_2, $\mathrm{H}^1(\mathbb{G}_{\mathbb{Q}}, \operatorname{Num} \overline{X}) = 0$.

Recall that $\operatorname{Pic} \overline{X}$ fits into an exact sequence

$$0 \longrightarrow \langle K_X\rangle \longrightarrow \operatorname{Pic} \overline{X} \longrightarrow \operatorname{Num} \overline{X} \longrightarrow 0.$$

Since $\mathrm{H}^1(G_{\mathbb{Q}}, \operatorname{Num} \overline{X}) = 0$, the long exact sequence from cohomology yields a surjective map $\mathrm{H}^1(G_{\mathbb{Q}}, \langle K_X\rangle) \to \mathrm{H}^1(G_{\mathbb{Q}}, \operatorname{Pic} \overline{X})$. By [Sko01, Thm. 6.1.2], we have

$$X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} = X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{im} \mathrm{H}^1(G_{\mathbb{Q}}, \langle K_X\rangle)} = \bigcup_{[\tau] \in \mathrm{H}^1(G_{\mathbb{Q}}, \langle K_X\rangle)} f^{\tau}\left(Y^{\tau}(\mathbb{A}_{\mathbb{Q}})\right).$$

By Proposition 5.2, $Y(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$, hence $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}_1} \neq \emptyset$, as desired.

5.4. **Transcendental Brauer elements.** In this section we will sketch a proof of the following result.

Theorem 5.13 ([BBM+]). *We have* $\operatorname{Br} X = \operatorname{Br}_1 X$, *in particular* $X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{Br}_1}$.

5.4.1. *General results.* Let S be a smooth proper k-rational surface and let $\pi\colon Y \to S$ be a smooth proper double cover. We will explain how π can be leveraged to understand certain Brauer classes on Y. Combining (2.3) with the functoriality of the Brauer group, we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
0 \longrightarrow & (\operatorname{Br} Y)[2] & \longrightarrow & (\operatorname{Br} \mathbf{k}(Y))[2] & \xrightarrow{(\partial_V)} & \bigoplus_{V \in Y^{(1)}} \mathrm{H}^1(\kappa(V), \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \\
& \uparrow \pi^* & & \uparrow \pi^* & & \uparrow \pi^* \\
0 \longrightarrow & (\operatorname{Br} S)[2] & \longrightarrow & (\operatorname{Br} \mathbf{k}(S))[2] & \xrightarrow{(\partial_V)} & \bigoplus_{V' \in S^{(1)}} \mathrm{H}^1(\kappa(V'), \frac{1}{2}\mathbb{Z}/\mathbb{Z})
\end{array}$$

We make two important observations:

- S is k-rational so $\operatorname{Br} S = \operatorname{Br}_0 S$, and
- $\mathbf{k}(Y)/\mathbf{k}(S)$ is a cyclic extension so $\ker \pi^* = \operatorname{Br} \mathbf{k}(Y)/\mathbf{k}(S)$ is completely understood (see (2.2)).

Proposition 5.14. *Let B denote the branch locus of π and let $Z_1, \ldots, Z_n$ be a basis for* $\operatorname{Pic} S$ *with $Z_i \neq B$ for all i. Assume that B is geometrically integral. If $\alpha \in \operatorname{Br} \mathbf{k}(S)$ such that $\pi^*\alpha \in \operatorname{Br} Y$, then there exists a $\beta \in \operatorname{Br} \mathbf{k}(S)$ such that $\partial_{V'}(\beta) = 0$ for all $V' \neq Z_i, B$ and $\pi^*\beta = \pi^*\alpha$.*

Sketch of proof. By the above commutative diagram, for any $V' \in S^{(1)}$ and any irreducible component $V \subset \pi^{-1}(V')$, we have

$$\partial_V(\pi^*\alpha) = \pi^*\partial_{V'}(\alpha).$$

Since $\pi^*\alpha \in \operatorname{Br} Y$, we must have that $\pi^*\partial_{V'}(\alpha) = 0$ for all $V' \in S^{(1)}$. Further, if $V' \neq B$ then the kernel of

$$\frac{\kappa(V')^\times}{\kappa(V')^{\times 2}} \cong \mathrm{H}^1\left(\kappa(V'), \frac{1}{2}\mathbb{Z}/\mathbb{Z}\right) \xrightarrow{\pi^*} \mathrm{H}^1\left(\kappa(V), \frac{1}{2}\mathbb{Z}/\mathbb{Z}\right) \cong \frac{\kappa(V)^\times}{\kappa(V)^{\times 2}}$$

is generated by $[g|_{V'}]$, where $g \in \mathbf{k}(S)^\times$ is such that $\sqrt{g}$ generates the extension $\mathbf{k}(Y)/\mathbf{k}(S)$ and $v_{V'}(g) = 0$. In particular, if $V' \neq B$, then there is a unique nontrivial element in the kernel of π^*.

Let D be the union of all curves $V' \in S^{(1)}$ such that $\partial_{V'}(\alpha) \neq 0$. Then there exist integers $m_1, \ldots, m_n$ and a function $f \in \mathbf{k}(S)^\times$ such that

$$\operatorname{div}(f) = D - m_1 Z_1 - \ldots m_n Z_n.$$

Then the quaternion algebra $\alpha' := (g, f)$ is such that $\partial_{V'}(\alpha') \neq 0$ for all components V' of D and $\partial_{V'}(\alpha') = 0$ for all $V' \neq Z_i, B$ and V' not contained in D. In addition, since g is a square in $\mathbf{k}(Y)^\times$, $\alpha' \in \ker \pi^*$ and in particular $\partial_{V'}(\alpha') \in \ker \pi^*$. Hence $\partial_{V'}(\alpha') = \partial_{V'}(\alpha)$ for all $V' \neq B, Z_i$. Thus $\beta := \alpha - \alpha'$ has the desired properties. ☐

Remark 5.15. Proposition 5.14 is sharp in the sense that we cannot enlarge the set of $V' \in Y^{(1)}$ for which $\partial_{V'}(\alpha) = \partial_{V'}(\beta)$. For instance, the classification of rational surfaces implies that there is a choice of $Z_1, \ldots, Z_n$ such that $S \setminus (Z_1 \cup \cdots \cup Z_n) \cong \mathbb{A}^2$ so $\operatorname{Br} S \setminus (Z_1 \cup \cdots \cup Z_n) = \operatorname{Br}_0 S$.

This proposition implies that if we care only about the unramified classes in the image of π^*, we may restrict to considering elements of $\operatorname{Br} U_S$, where $U_S := S \setminus (B \cup Z_1 \cup \cdots \cup Z_n)$. Then we may use similar arguments to characterize the elements $\alpha \in \operatorname{Br} U_S$ such that $\pi^*\alpha \in \operatorname{Br}_0 Y$ or such that $\pi^* \in \operatorname{Br}_1 Y$.

Proposition 5.16 ([IOOV, Thm. 1.3] for $\operatorname{Br}_1 Y$ and [BBM$^+$, Prop. 2.8] for $\operatorname{Br}_0 Y$)**.** *If $\alpha \in \operatorname{Br} U_S$ is such that $\pi^*\alpha \in \ker(\operatorname{Br} \mathbf{k}(Y) \to \operatorname{Br} \mathbf{k}(\overline{Y}))$, then there exists a divisor $D \subset X^{\text{sep}} \setminus \pi^{-1}(B)$, integers $m_1, \ldots, m_n$, and a function $f \in \mathbf{k}(S_{k^{\text{sep}}})^\times$ such that $\partial_B(\alpha) = [f|_B]$ and*

$$\operatorname{div}(f) = \pi_*(D) - m_1 Z_1 - \ldots m_n Z_n.$$

If $\pi^*\alpha \in \operatorname{Br}_0 Y$, then we may further assume that $D \subset X \setminus \pi^{-1}(B)$ and $f \in \mathbf{k}(S)^\times$.

Building on ideas from [CV14a], given any function ℓ on B contained in the subgroup

$$\mathbf{k}(B)_{\mathcal{E}} := \{\ell \in \mathbf{k}(B) : \operatorname{div}(\ell) \bmod 2 \in \langle Z_1|_B, Z_2|_B, \ldots, Z_n|_B \subset \operatorname{Div}(B)/2\operatorname{Div}(B)\rangle\},$$

we may construct a central simple algebra α_ℓ such that $\alpha_\ell \in \operatorname{Br} U_S$ and $\partial_B(\alpha_\ell) = [\ell]$ [BBM$^+$, §2.2]. Further, this construction induces a homomorphism

$$\beta\colon \frac{\mathbf{k}(B)_{\mathcal{E}}^\times}{k^\times \mathbf{k}(B)^{\times 2}} \longrightarrow \frac{\operatorname{Br} Y}{\operatorname{Br}_0 Y}, \quad [\ell] \mapsto \pi^*\alpha_\ell.$$

Proposition 5.16 characterizes the kernel of β and $\beta^{-1}(\operatorname{Br}_1 Y)$ (see [BBM$^+$, Thms. 2.1 and 2.2] for precise statements). Additionally, if k is separably closed, then by [CV14a, Thm. I] β surjects onto $(\operatorname{Br} Y)[2]$.

5.4.2. *Brauer classes on Y.* Recall that there is a 2-to-1 morphism $\pi\colon Y \to S$, where S is the del Pezzo surface given by

$$xy + 5z^2 - s^2 = x^2 + 3xy + 2y^2 - s^2 + 5t^2 = 0.$$

As S is rational over $K_0 := \mathbb{Q}(\sqrt{5}, i, \sqrt{-2+2\sqrt{2}})$, we may apply the results of the previous section. The branch curve of π is a smooth genus 5 curve given by $V(12x^2+111y^2+13z^2) \subset S$. Note that $\pi^{-1}(B)_{\mathrm{red}} \cong B$ is fixed by σ; write $\widetilde{B}$ for the quotient B/σ. The curve $\widetilde{B}$ is a (geometrically) hyperelliptic genus 3 curve.

Let K_1/K_0 be the splitting field of the Weierstrass points of $\widetilde{B}$ and let $Z_1, \dots, Z_6 \subset S$ be rational curves defined over K_0 that generate $\operatorname{Pic} \overline{S}$ and such that $U := S \setminus (Z_1 \cup \cdots \cup Z_6) \cong \mathbb{A}^2$.[3] We claim that the following facts hold:

(1) $\operatorname{Br}_1 \pi^{-1}(U_{K_1}) / \operatorname{Br} K_1$ is generated by classes of the form $\beta(\ell)$ for $\ell \in \mathbf{k}(\widetilde{B}_{K_1})$ such that $\operatorname{div}(\ell) \in 2\operatorname{Div}(\widetilde{B})$ [BBM+, Thm. 2.2],
(2) Over K_1, β is injective [BBM+, §5, Appendix A],
(3) There exists an element $\tilde{\ell} \in \mathbf{k}(\widetilde{B}_{K_1})$ such that $\operatorname{div}(\tilde{\ell}) \in 2\operatorname{Div}(\widetilde{B})$ and $\beta(\tilde{\ell}) \otimes_{K_1} \overline{\mathbb{Q}}$ generates $f^* \operatorname{Br} \overline{X}$ [BBM+, Prop. 4.1], and
(4) For all $\ell \in (\mathbf{k}(\widetilde{B}_{K_1})_{\mathcal{E}} / K_1^{\times} \mathbf{k}(B)^{\times 2})^{\operatorname{Gal}(K_1/K_0)}$ such that $\operatorname{div}(\ell) \in 2\operatorname{Div}(\widetilde{B})$, $\beta(\ell) \otimes_{K_1} \overline{\mathbb{Q}}$ does *not* generate $f^* \operatorname{Br} \overline{X}$ [BBM+, §5, Appendix A].

5.4.3. *Sketch of proof of Theorem 5.13.* Using a criterion of Beauville [Bea09], Várilly-Alvarado and I computed that the map $f^*\colon \operatorname{Br} \overline{X} \to \operatorname{Br} \overline{Y}$ is injective. Thus if $\operatorname{Br} X \neq \operatorname{Br}_1 X$, then there exists an element $\alpha_Y \in \operatorname{Br} Y$ such that $\alpha_Y \otimes_k \overline{\mathbb{Q}}$ generates $f^* \operatorname{Br} \overline{X}$. Thus, we can reduce to working on the K3 surface Y.

Let K_1, K_0 and U be as above. Assume that there exists an $\alpha_Y \in \operatorname{Br} Y$ such that $\alpha_Y \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ generates $f^* \operatorname{Br} \overline{X}$. Then $\alpha_Y \in \left(\frac{\operatorname{Br}(\pi^{-1}(U_{K_1}))}{\operatorname{Br}_0(\pi^{-1}(U_{K_1}))}\right)^{\operatorname{Gal}(K_1/K_0)}$. Since $\alpha_Y \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ generates $f^* \operatorname{Br} \overline{X}$, by (3) there exists a $\tilde{\ell} \in \mathbf{k}(\widetilde{B}_{K_1})$ with $\operatorname{div}(\tilde{\ell}) \in 2\operatorname{Div}(\widetilde{B})$ such that $\alpha_Y \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \beta(\tilde{\ell}) \otimes_{K_1} \overline{\mathbb{Q}}$ or equivalently such that $\alpha_Y - \beta(\tilde{\ell}) \in \operatorname{Br}_1 \pi^{-1}(U_{K_1})$. Then by (1), $\alpha_Y = \beta(\ell)$ for some $\ell \in \mathbf{k}(\widetilde{B})$ such that $\operatorname{div}(\ell) \in 2\operatorname{Div}(\widetilde{B})$. Further, since α_Y is fixed by $\operatorname{Gal}(K_1/K_0)$, ℓ must be fixed by $\operatorname{Gal}(K_1/K_0)$ modulo $(\ker \beta) K_1^{\times} \mathbf{k}(B)^{\times 2}$. This gives a contradiction by (2) and (4).

6. Project descriptions

I have tried to come up with projects that are exploratory in nature. In complete generality, many of these projects may be very ambitious. But I believe these problems are already interesting in some examples, so I encourage you to start there! For each project, I have listed some suggested references, as well as my best guess of what sorts of tools will be useful. **However**, I have not solved these problems, so I do not actually know what will be used!

6.1. **Double covers of surfaces and 2-torsion Brauer classes.** Let $\pi\colon X \to S$ be a smooth double cover of a rational geometrically ruled surface. If k is separably closed and has characteristic different from 2, the pullback map $\pi^*\colon \operatorname{Br} \mathbf{k}(S) \to \operatorname{Br} \mathbf{k}(X)$ surjects onto the subgroup $\operatorname{Br} X[2]$ [CV14a]. Since every non-constant Brauer class on S is ramified somewhere, this means that we may study unramified 2-torsion Brauer classes on X by studying ramified Brauer classes on S, which is generally easier. The proof relies heavily

[3] Such a choice of curves exist, see [BBM+, §3]

on the separably closed hypothesis. If the ground field is *not* separably closed, does $\operatorname{im}\pi^*$ still contain $\operatorname{Br} X[2]$? Theorem 2.2 in [BBM$^+$] shows that the 2-torsion of algebraic Brauer group of an open set in X is contained in the image of π^* modulo some assumptions. Are the assumptions necessary? Can the result be extended to all of X?

Kresch and Tschinkel have computed the Brauer group of certain diagonal degree 2 del Pezzo surfaces [KT04]. Since any degree 2 del Pezzo surface is a double cover of $\mathbb{P}^2$, you could play with some examples from this paper.
Suggested references: [BBM$^+$, Sko, CV14a, KT04]
Preferable background: Galois cohomology, Picard groups, cyclic covers, residue maps

6.2. **Relationship between $\operatorname{Br} Y$ and $\operatorname{Br} Y^\tau$ where Y^τ is a twist of Y.** If k is a global field, then the étale-Brauer set of X is

$$X(\mathbb{A}_k)^{\text{et},\text{Br}} := \bigcap_{\substack{G \text{ finite étale} \\ f\colon Y\to X, \\ \text{a } G-\text{torsor}}} \bigcup_{[\tau]\in \mathrm{H}^1(k,G)} f^\tau(Y^\tau(\mathbb{A}_k)^{\operatorname{Br} Y^\tau}).$$

To aid in computation of $X(\mathbb{A}_k)^{\text{et},\text{Br}}$, it would be desirable to determine the strongest possible relationship between $\operatorname{Br} Y$ and $\operatorname{Br} Y^\tau$. You might start with considering the algebraic Brauer group and trying to understand how the action of G_k on $\operatorname{Pic}\overline{Y}$ differs from the action of G_k on $\operatorname{Pic}\overline{Y}^\tau$. Perhaps if p is coprime to the order of G, then $\operatorname{Br}_1 Y[p] \cong \operatorname{Br}_1 Y^\tau[p]$?
Suggested references: [Sko01, Poo14, Ier10]
Preferable background: Twists, étale cohomology, Galois cohomology

6.3. **Central simple algebra representatives for p-torsion transcendental Brauer classes with $p > 2$.** Let k be a separably closed field of characteristic different from p, let $\pi\colon X \to \mathbb{P}^2$ be a p-cyclic cover over k, and let C be the branch curve of π. Then there is an action of $\mathbb{Z}[\zeta]$ on $\operatorname{Br} X$ and $\operatorname{Br} X[1-\zeta] \subset \operatorname{im}\pi^*\colon \operatorname{Br} U \to \operatorname{Br} \mathbf{k}(X)$, where U is the open set of $\mathbb{P}^2$ obtained by removing B and a fixed line L [IOOV]. Furthermore, the elements of $(\operatorname{Br} U)[p]$ are characterized by their residues on C. Thus, if one can construct central simple algebras over $\mathbf{k}(\mathbb{P}^2)$, unramified on U and with prescribed residue at C, then one can construct representatives for the $(1-\zeta)$-torsion in $\operatorname{Br} X$.

Let $P \in L\setminus(B\cap L)$, let $S' = \operatorname{Bl}_P(S)$, and let $X' = \operatorname{Bl}_{\pi^{-1}(P)}(X)$. Then S' is a ruled surface, with the ruling induced by the projection map $\mathbb{P}^2 \setminus P \to \mathbb{P}^1_T$, and there exists a double cover morphism $f'\colon X' \to S'$ whose branch locus is the strict transform of B. The generic fiber of the composition $X' \to S' \to \mathbb{P}^1$ is a curve C, with a model of the form $y^2 = cf(x)$, where f is a monic polynomial over $k(T)$ and $c \in k(T)^{\times 2}$. Then the results and proofs of [CV14a] show that for any $\ell \in \mathbf{k}(B)^\times$, in the case $p = 2$,

$$\mathcal{A}_\ell := \operatorname{Cor}_{\mathbf{k}(B)/k(T)}\left((x-\alpha,\ell)_2\right) \in \operatorname{Br}\mathbb{P}^2 \setminus (B\cup L).$$

Further, if $p = 2$ then $\partial_B(\mathcal{A}_\ell) = [\ell]$ [BBM$^+$, Lemma 2.3].

For $p > 2$, does there exist an integer $1 \le i \le p-1$ such that

$$\mathcal{A}_\ell := \operatorname{Cor}_{\mathbf{k}(B)/k(T)}\left((x-\alpha,\ell^i)_2\right) \in \operatorname{Br}\mathbb{P}^2 \setminus (B\cup L)$$

and $\partial_B(\mathcal{A}_\ell) = [\ell]$?

If so, then algebras of the form $\pi^*\mathcal{A}_\ell$ would generate the $(1-\zeta)$-torsion of $\operatorname{Br} X$. If this does hold, then one could try to generalize [BBM$^+$, §2] to p-cyclic covers of $\mathbb{P}^2$.
Suggested references: [CV14a, IOOV, BBM$^+$]

Preferable background: Blow-ups, Picard groups of rational surfaces, Computation of residues

6.4. **The cokernel of** $\operatorname{Br}_1 X \to \mathrm{H}^1(G_k, \operatorname{Pic} \overline{X})$**.** By the Hochschild-Serre spectral sequence, we have an exact sequence

$$\operatorname{Br}_1 X \to \mathrm{H}^1(G_k, \operatorname{Pic} \overline{X}) \to \mathrm{H}^3(G_k, \mathbb{G}_m).$$

If k is a global field, then $\mathrm{H}^3(G_k, \mathbb{G}_m) = 0$ so every element of $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X})$ lifts to an algebraic Brauer class on X. If k is an arbitrary field, this may no longer hold. For example, Uematsu showed that if $k = \mathbb{Q}(\zeta_3, a, b, c)$ where a, b, c are independent transcendentals, and X is a cubic surface, then the map $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X}) \to \mathrm{H}^3(k, \mathbb{G}_m)$ can be nonzero [Uem14].

Let X be a del Pezzo surface of degree 4, i.e., a smooth intersection of 2 quadrics in $\mathbb{P}^4$. We may associate to X a pencil of quadrics $V \to \mathbb{P}^1$. A general fiber of V is rank 5; there is a reduced degree 5 subscheme $\mathcal{S} \subset \mathbb{P}^1$ where the quadrics have rank strictly less than 5. There are necessary and sufficient conditions in terms of these quadrics of lower rank for the existence of a nontrivial element of $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X})$ [VAV14]. If, in addition, certain degenerate quadrics have a rational point (over their field of definition), then there is a construction which lifts a nontrivial element of $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X})$ to an algebraic Brauer class on X [VAV14].

This raises the question of whether this condition of having a rational point is necessary, i.e., does there exist a field k and a degree 4 del Pezzo surface over k with the map $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X}) \to \mathrm{H}^3(k, \mathbb{G}_m)$ nontrivial?

The map $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X}) \to \mathrm{H}^3(G_k, \mathbb{G}_m)$ is the composition of two boundary maps:

(1) the map $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X}) \to \mathrm{H}^2(G_k, \mathbf{k}(\overline{X})^\times / \overline{k}^\times)$ coming from the short exact sequence

$$0 \to \mathbf{k}(\overline{X})^\times / \overline{k}^\times \to \operatorname{Div} \overline{X} \to \operatorname{Pic} \overline{X} \to 0,$$

and

(2) the map $\mathrm{H}^2(G_k, \mathbf{k}(\overline{X})^\times / \overline{k}^\times) \to \mathrm{H}^3(G_k, \overline{k}^\times)$ coming from the short exact sequence

$$0 \to \overline{k}^\times \to \mathbf{k}(\overline{X})^\times \to \mathbf{k}(\overline{X})^\times / \mathbf{k}^\times \to 0.$$

If all rank 4 quadrics in the pencil have a k-point, then the first boundary map $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X}) \to \mathrm{H}^2(G_k, \mathbf{k}(\overline{X})^\times / \overline{k}^\times)$ is identically 0. If the rank 4 quadrics fail to have a k-point, then it is at least not obvious that the first boundary map is 0.

Suggested references: [VAV14, Uem14]

Preferable background: Galois cohomology, (some) geometry of pencils of quadrics

6.5. **Brauer groups of del Pezzo surfaces.** Let X be a del Pezzo surface over a global field k. Since X is geometrically rational, $\operatorname{Br} X = \operatorname{Br}_1 X$. Additionally, there are finitely many possibilities for $\operatorname{Br}_1 X / \operatorname{Br} k$ [Cor07, Thm. 4.1]. If we fix the degree of X and assume that X is minimal, i.e., there are no Galois invariant subsets of pairwise skew (-1)-curves, what are the possibilities for $\operatorname{Br}_1 X / \operatorname{Br} k$? Alternatively (or in addition!), for each possible isomorphism class of $\operatorname{Br}_1 X / \operatorname{Br} k$, you could construct a del Pezzo surface X that has that particular Brauer group. For instance, you could try to construct a del Pezzo surface (necessarily of degree 1), with an order 5 Brauer class?

Suggested references: [VA13, VA08, Cor07, Car11]

Preferable background: Galois cohomology, blow-ups, del Pezzo surfaces, (maybe) root systems

6.6. **2-torsion transcendental classes on diagonal quartics.** Let k be a field of characteristic different from 2 and let $X \subset \mathbb{P}^3$ be a diagonal quartic surface, i.e., X is given by

$$ax^4 + by^4 + cz^4 + dw^4 = 0$$

for some $a, b, c, d \in k^\times$. Note that X is a double cover of a del Pezzo surface of degree 2 (in four different ways in fact). For instance, consider $X \to V(ax^4 + by^4 + cz^4 + dw^2) \subset \mathbb{P}(2, 1, 1, 1)$. Using this double cover structure, Skorobogatov proved that $\operatorname{Br} \overline{X}$ has a Galois invariant element of order 2 [Sko, Prop. 3.6]. Skorobogatov asks:

(1) When does the image of the natural map $\operatorname{Br} X \to \operatorname{Br} \overline{X}$ contain an element of order 2?
(2) What is the full Galois action on $(\operatorname{Br} \overline{X})[2]$

Evis Ieronymou studied the arithmetic of $X/\mathbb{Q}(i, \sqrt[4]{2})$ in the case that $a = d = -b = -c = 1$. In doing so, he computed representatives for $(\operatorname{Br} \overline{X})[2]$ [Ier10, Thm. 3.11]. He uses this to study the Brauer group of X over certain number fields [Ier10, §§4,5]. He proves that if $2 \notin \langle -1, 4, b/a, c/a, d/a\rangle \subset \mathbb{Q}^\times/\mathbb{Q}^{\times 4}$, then $(\operatorname{Br} X)[2] \to \operatorname{Br} \overline{X}$ is the 0 map. If 2 is in this subgroup, then the above questions are not answered. The methods of [BBM+] should give at least partial answers to the above questions.

Remark 6.1. The odd torsion is completely understood [IS15, ISZ11].

Suggested references: [Sko, Ier10, BBM+]
Preferable background: Computation of residues, Picard groups of K3 surfaces and del Pezzo surfaces.

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