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Weak sharp minima revisited, part II: application to linear regularity and error bounds

We dedicate this paper to our friend and mentor Terry Rockafellar on the occasion of his 70th birthday. He has been our guide in mathematics as well as in the backcountry and waterways of the Olympic and Cascade mountains.

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Abstract. The notion of weak sharp minima is an important tool in the analysis of the perturbation behavior of certain classes of optimization problems as well as in the convergence analysis of algorithms designed to solve these problems. It has been studied extensively by several authors. This paper is the second of a series on this subject where the basic results on weak sharp minima in Part I are applied to a number of important problems in convex programming. In Part II we study applications to the linear regularity and bounded linear regularity of a finite collection of convex sets as well as global error bounds in convex programming. We obtain both new results and reproduce several existing results from a fresh perspective.

Key words. weak sharp minima – local weak sharp minima – boundedly weak sharp minima – recession function – recession cone – linear regularity – additive regularity – constraint qualification – error bounds

1. Introduction

We continue our study of *weak sharp minima* focusing on applications to linear regularity [4, 5, 26] and global error bounds for convex inclusions [2, 10, 12, 18, 20–22, 24, 23, 25, 28, 30, 40, 39]. The history and motivation for the study of weak sharp minima is reviewed in Part I of this work where we also established much of the theoretical foundations in the infinite dimensional setting. Weak sharp minima were first studied by Polyak in [29] as a set-valued extension to the notion of a sharp minima. Ferris [15] introduced the name *weak sharp minima* and developed a number of basic properties. Part I builds on [9] by extending the results to infinite dimension and by making a number of refinements that broaden the range of applications. We use the new tools developed in Part I to examine the relationship between weak sharp minima, linear regularity, and global error bounds. This is done by equating the linear regularity property or the existence of a global error bound with the existence of a set of weak sharp minima for an underlying convex function, and then apply the results of Part I. In this way results on

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linear regularity and global error bounds are reduced to properties of the subdifferential of an appropriately chosen convex function. For some very recent work on weak sharp minima closely related to work in Part I, we direct the reader to the very nice paper by Zălinescu [37].

Connections to the linear regularity are examined in Section 3. Our study is motivated by the recent paper of Bauschke, Borwein, and Li [5], and are also related to further refinements appearing recently in [26, 40]. We recover many of the basic facts about linear regularity from corresponding facts about weak sharp minima, and obtain a number of new sufficient conditions for linear and bounded linear regularity in the infinite dimensional setting. A new concept for finite collections of cones called *additive regularity* is introduced and used to obtain conditions for establishing the strong CHIP property and Jameson's property (G).

In Section 4, we focus on error bounds for nondifferentiable systems of convex inequalities. Here we are motivated by the recent paper by Lewis and Pang [22]. These results are also related to recent work appearing in [40] and [39]. We recover the results of Lewis and Pang on the characterization of the existence of global error bounds and obtain several new characterizations as well. Following Bauschke, Bowein, and Li [5], we further investigate the link between the existence of a global error bound and the linear regularity of the underlying level sets. In particular, we improve Bauschke, Bowein, and Li's result [5, Theorem 8] establishing the linear regularity of underlying level sets under the weak Slater condition and the Auslender-Crouzeix [2] asymptotic constraint qualification.

In Section 2 we recall two key results from Part I that form the foundation for our investigations.

2. Basic results and notation

Let X be a normed linear space, and consider the nonempty closed convex sets $\tilde{S} \subset S \subset X$ and the lower semi-continuous convex function $f\colon X \mapsto \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We assume that $S \cap \text{dom}\,(f) \neq \emptyset$ where $\text{dom}\,(f) = \{x \in X \mid f(x) < \infty\}$. The set $\tilde{S} \subset X$ is said to be a set of *weak sharp minima* for the function f over the set S with modulus $\alpha > 0$ if

$$f(\bar{x}) + \alpha \text{dist}\,(x \mid \tilde{S}) \leq f(x) \quad \text{for all } \bar{x} \in \tilde{S} \text{ and } x \in S, \tag{1}$$

where $\text{dist}\,(x \mid \tilde{S}) = \inf_{\bar{x} \in \tilde{S}} \|x - \bar{x}\|$, and $\|\cdot\|$ is the norm on X. Since $S \cap \text{dom}\,(f) \neq \emptyset$ we have $\tilde{S} = \arg\min_S f \subset \text{dom}\,(f)$, where

$$\arg\min_S f = \left\{ x \in S \;\middle|\; f(x) = \min_{y \in S} f(y) \right\}.$$

In Part I we provide several different characterizations of weak sharp minima. Of these, we focus on only one for the applications studied in this paper. We state this characterization using standard notation. An explanation of this notation is given at the end of this section.

Theorem 1. [8, Theorem 2.3] *Let f, S, and $\tilde{S}$ be as in (1), and assume that the addition formula*

$$\partial(f + \psi_S)(x) = \mathrm{cl}^* \left(\partial f(x) + N_S(x)\right), \tag{2}$$

holds for all $x \in \tilde{S}$. Let $\alpha > 0$. Then the set $\tilde{S}$ is a set of weak sharp minima for the function f over the set $S \subset X$ with modulus α if and only if the normal cone inclusion

$$\alpha \mathbb{B}^\circ \cap N_{\tilde{S}}(x) \subset \mathrm{cl}^* \left(\partial f(x) + N_S(x)\right) \tag{3}$$

holds for all $x \in \tilde{S}$.

Remark 1. The significance of the weak calculus formula (2) is illustrated in Part I by example.

It is shown in Part I that the normal cone inclusion (3) can be decomposed into two independent conditions. These conditions play a pivotal role in connecting the notion of weak sharp minima to a number of related ideas in the literature.

Lemma 1. [8, Lemma 3.1] *Let the basic assumptions of Theorem 1 hold. Given $x \in \tilde{S}$, we have*

$$\alpha \mathbb{B}^\circ \cap N_{\tilde{S}}(x) \subset \mathrm{cl}^* \left(\partial f(x) + N_S(x)\right) \tag{4}$$

if and only if

$$\begin{gathered} \mathrm{cone}\left(\mathrm{cl}^* \left(\partial f(x) + N_S(x)\right)\right) = N_{\tilde{S}}(x) \qquad \textit{and} \\ \alpha \mathbb{B}^\circ \cap \left[\mathrm{cone}\left(\mathrm{cl}^* \left(\partial f(x) + N_S(x)\right)\right)\right] \subset \mathrm{cl}^* \left(\partial f(x) + N_S(x)\right). \end{gathered} \tag{5}$$

In addition, if the set $\partial f(x) + N_S(x)$ is weak* closed, then

$$\mathrm{cone}\left(\mathrm{cl}^* \left(\partial f(x) + N_S(x)\right)\right) = \mathrm{cone}\left(\partial f(x)\right) + N_S(x).$$

The notation that we employ is consistent with that used in Part I, and is for the most part the same as that in [1, 32, 33]. A partial list is provided below for the reader's convenience.

Denote the dual space of X by X^*. When X is endowed with the weak topology and X^* with the weak* topology then the spaces X and X^* are said to be *paired in duality* by the continuous bi-linear form $\langle x^*, x\rangle = x^*(x)$ defined on $X^* \times X$ [33]. Denote the norm on X^* by $\|\cdot\|_\circ$: $\|z\|_\circ = \sup_{x \in \mathbb{B}} \langle z, x\rangle$, where $\mathbb{B} = \{x \in X \mid \|x\| \le 1\}$ is the unit ball in X. We will use the notation $\mathbb{B}$ for the unit ball of whatever space we are discussing. If there is a possibility of confusion, we will write $\mathbb{B}_Z$ for the unit ball in the normed linear space Z. Given a set C in either X or X^*, the set $\mathrm{cl}\,(C)$ is the closure of this set in the norm topology, and given a set E in X^*, the set $\mathrm{cl}^*\,(E)$ is the closure in the weak* topology.

For a nonempty subset C of any normed linear space Y, denote the indicator function of C and the support function of C by $\psi_C(\cdot)$ and $\psi_C^*(\cdot)$, respectively. Thus, in particular, $\|z\|_o = \psi_{\mathbb{B}}^*(z)$. The barrier and recession cones of a convex set C are $\mathrm{bar}\,(C) = \mathrm{dom}\,\left(\psi_C^*(\cdot)\right)$ and $C^\infty = \{d \mid x + td \in C \;\forall x \in C,\ t > 0\}$, respectively. The norm-topology interior of C is $\mathrm{int}\,(C)$, and the boundary of C is $\mathrm{bdry}\,(C) = \mathrm{cl}\,(C) \setminus \mathrm{int}\,(C)$.

When Y is finite dimensional, ri (C) is the interior of C relative to the smallest affine set containing C. The cone generated by C is cone $(C) = \cup_{\lambda \geq 0}\{\lambda C\}$.

For a closed convex set C in X, an *extreme point* of C is any point in the convex set that cannot be represented as the convex combination of two other points in C. Define the *projection* of a point $x \in X$ onto the set C, denoted $P(x \mid C)$, as the set of all points in C that are closest to x as measured by the norm $\|\cdot\|$: $P(x \mid C) = \{ y \in C \mid \|x - y\| = \text{dist}\,(x \mid C) \}$. For nonempty sets $C \subset X$ and $E \subset X^*$, the polar of C and E are given by the sets $C^\circ = \{x^* \in X^* \mid \langle x^*, x\rangle \leq 1 \ \forall x \in C\}$, $E^\circ = \{x \in X \mid \langle x^*, x\rangle \leq 1 \ \forall x^* \in E\}$, respectively. Thus, in particular, $\mathbb{B}^\circ \subset X^*$ is the unit ball associated with the dual norm $\|\cdot\|_o$. If either C or E is a subspace, we also write $C^\circ = C^\perp$ and $E^\circ = E^\perp$. For a nonempty closed convex set C in X, and $x \in C$, define the tangent cone to C at x by $T_C(x) = \text{cl}\left(\bigcup_{t>0} \frac{C-x}{t}\right)$. The normal cone to C at x is given by $N_C(x) = T_C(x)^\circ$. It is easy to see that $N_C(x) = \{x^* \in X^* | \langle x^*, y - x\rangle \leq 0, \quad \text{for any } y \in C\}$.

3. Linear regularity

Linear regularity has been extensively studied in [4, 5], where its importance for the study of algorithms is examined. We discuss this notion for two reasons. First, it is an example that illustrates the power of Theorem 1 in an important application and second, there is a close connection between linear regularity and the important concepts of metric regularity and error bounds for convex inequalities.

Definition 1. *Let $\{ C_i \mid i = 1, \ldots, N \}$ be a collection of nonempty closed convex subsets of the normed linear space X and suppose that the convex set $C = \bigcap_{i=1}^N C_i$ is nonempty. The collection $\{ C_i \mid i = 1, \ldots, N\}$ is said to be linearly regular if there exists $\alpha > 0$ such that*

$$\alpha \text{dist}\,(y \mid C) \leq \max_{i=1,\ldots,N} \text{dist}\,(y \mid C_i) \tag{6}$$

for all $y \in X$. The collection $\{ C_i \mid i = 1, \ldots, N \}$ is said to be boundedly linearly regular if, for every bounded subset $D \subset X$, there exists $\alpha > 0$ such that (6) holds for all $y \in D$.

Let $\|\cdot\|$ be a norm on $\mathbb{R}^N$ that is monotone with respect to the cone $\mathbb{R}^N_+$, i.e. if $u, v \in \mathbb{R}^N$ satisfy $0 \leq u_i \leq v_i$, $i = 1, 2, \ldots, N$, then $\|u\| \leq \|v\|$. For example, the l_p norms are monotone with respect to $\mathbb{R}^N_+$. Define $\rho\colon X \mapsto \mathbb{R}$ by

$$\rho(x) = \|F(x)\|\,, \tag{7}$$

where $F\colon X \mapsto \mathbb{R}^N$ has component functions $F_i(x) = \text{dist}\,(y \mid C_i)$, and the norm is monotone. Using the convexity of each of the functions F_i and the monotonicity of the norm, it is straightforward to show that ρ is convex. We are interested in conditions that characterize when the set C is a set of weak sharp minima for the function ρ. Observe that if $\|\cdot\| = \|\cdot\|_\infty$, then C is a set of weak sharp minima for the function ρ if and only if the collection $\{ C_i \mid i = 1, \ldots, N \}$ is linearly regular. Indeed, if $\|\cdot\|_a$ and $\|\cdot\|_b$ are two monotone norms on $\mathbb{R}^N$ with ρ_a and ρ_b corresponding to the function defined in (7) for these norms, respectively, then, due to the equivalence of norms, we know that if C is a set of weak sharp minima for ρ_a, then it is a set of weak sharp minima for ρ_b as well. That is, the collection $\{ C_i \mid i = 1, \ldots, N \}$ is linearly regular if and only if C is a set of weak sharp minima for the function ρ regardless of the choice of monotone norm.

Theorem 2. *Let $\rho\colon X \mapsto \mathbb{R}$ be as defined above. Then the collection of sets $\{C_i i = 1, \ldots, N\}$ is linearly regular if and only if there is an $\bar{\alpha} > 0$ such that*

$$\bar{\alpha}\mathbb{B}^\circ \cap N_C(x) \subset \partial\rho(x) \quad \forall\, x \in C, \tag{8}$$

or equivalently, there is an $\alpha > 0$ such that

$$\alpha\mathbb{B}^\circ \cap N_C(x) \subset \sum_{i=1}^{N}(\mathbb{B}^\circ \cap N_{C_i}(x)) \tag{9}$$

for all $x \in C$. In addition, the inclusion (9) is equivalent to the pair of conditions

$$N_C(x) = \sum_{i=1}^{N} N_{C_i}(x), \quad \textit{and} \tag{10}$$

$$\alpha\mathbb{B}^\circ \cap \left(\sum_{i=1}^{N} N_{C_i}(x)\right) \subset \sum_{i=1}^{N}(\mathbb{B}^\circ \cap N_{C_i}(x)). \tag{11}$$

Proof. We apply Theorem 1 to the function ρ defined in (7). First observe that the sets S and $\tilde{S}$ in Theorem 1 are given by X and C, respectively, so that the hypotheses of Theorem 1 are satisfied. Thus the inclusion (8) follows immediately from Theorem 1.

As has already been observed, the function ρ has the set C as a set of weak sharp minima if and only if the function $\rho_1(x) = \|F(x)\|_1 = \sum_{i=1}^{N} \operatorname{dist}(x \mid C_i)$ has C as a set of weak sharp minima. Therefore, the condition (8) is equivalent to the condition $\alpha\mathbb{B}^\circ \cap N_C(x) \subset \partial\rho_1(x)$. However, given x in C, we have from [8, Theorem A.1, Part 5] that $\partial\operatorname{dist}(x \mid C_i) = \mathbb{B}^\circ \cap N_{C_i}(x)$ for each $i = 1, 2, \ldots, N$. Hence, from [14, Proposition 5.6, page 26], $\partial\rho_1(x) = \sum_{i=1}^{N} \partial\operatorname{dist}(x \mid C_i)$, whereby the equivalence of (8) and (9) are established.

To obtain the equivalence of (9) with (10) and (11), we apply Lemma 1. This lemma states that (9) is equivalent to the two statements

$$N_C(x) = \operatorname{cone}\Big(\sum_{i=1}^{N}(\mathbb{B}^\circ \cap N_{C_i}(x))\Big),$$

$$\sum_{i=1}^{N}(\mathbb{B}^\circ \cap N_{C_i}(x)) \supset \alpha\mathbb{B}^\circ \cap \operatorname{cone}\Big(\sum_{i=1}^{N}(\mathbb{B}^\circ \cap N_{C_i}(x))\Big).$$

The equivalence of (9) to the pair of conditions (10) and (11) follows immediately from the simple identity $\operatorname{cone}\left(\sum_{i=1}^{N}(\mathbb{B}^\circ \cap N_{C_i}(x))\right) = \sum_{i=1}^{N} N_{C_i}(x)$. □

Remark 2. The equivalence of the linear regularity of $\{ C_i \mid i = 1, \ldots, N \}$ and the conditions (10) and (11) has been shown in [26, Theorem 4.2] in the Banach space setting. In [26, Theorem 4.2] the focus is on ℓ^p norms, but, as the discussion preceeding the Theorem shows, any monotone norm will do. In [40, Theorem 3.5] this result is generalized to possibly infinite collections of sets, $\{ C_i \mid i \in I \}$ with I arbitrary.

Theorem [8, Theorem 2.3] can be used to derive a number of other characterizations of linear regularity, however, the characterizations given in Theorem 2 are particularly significant due to their connection to the existing literature on this subject. Condition (10) is known as the *strong conical hull intersection property*, or strong CHIP [19], while condition (11) is known as *Jameson's property (G)* for the cones $N_{C_i}(x)$, $i = 1, \dots, N$ [19]. Both of these conditions are extensively studied in the literature [5, 19, 39]. Conditions assuring that strong CHIP (10) holds have long been known in the convex analysis literature. We give the two most well-known in our next proposition. Further conditions yielding the strong CHIP property can be found in Theorem 7 to follow.

Proposition 1. *Let $C,\ C_i,\ i = 1, \dots, N$ be as in Definition 1. Strong CHIP (10) holds at every point of C under either of the following two conditions hold:*

1. *[33, Theorem 20] There is an $i_0 \in \{1, \dots N\}$ such that $\left(\bigcap_{i=1,\, i\neq i_0} \operatorname{int}\ (C_i)\right) \cap C_{i_0} \neq \emptyset$.*
2. *[32, Corollary 23.8.1] The space X is finite dimensional and there is an integer $0 \le k_0 \le N$ such that $(\bigcap_{i=1}^{k_0} \operatorname{ri}\ (C_i)) \cap (\bigcap_{i=k_0+1}^{N} C_i) \neq \emptyset$ and C_i is polyhedral for $i = k_0 + 1, \dots, N$.*

Simple conditions assuring that (11) holds can be derived from [8, Lemma 3.4, Part 2]. Two useful notions in this context are that of a *base* of a convex cone and pointed cones [6, 16].

Definition 2. *Let K be a non-empty cone in a topological vector space.*

(i) The cone K is said to be pointed if $K \cap (-K) = \{0\}$.
(ii) Further assume that K is convex. A convex set $B \subset K$ is said to be a base for K if $0 \notin \operatorname{cl}\ (B)$ and $\operatorname{cone}(B) = K$.

In finite dimensions, a nonempty closed convex cone $K \neq \{0\}$ is pointed if and only if it has a bounded base [6, Page 60]. The following lemma helps us use these notions to establish inclusion (11).

Lemma 2. *Let $\mathcal{K}_i \subset X^*,\ i = 1, \dots, N$ be a collection of weak* closed convex cones and set*

$$\mathcal{K} = \sum_{i=1}^{N} \mathcal{K}_i, \quad B = \operatorname{co}\left(\cup_{i=1}^{N} (\operatorname{bdry}(\mathbb{B}^\circ) \cap \mathcal{K}_i)\right), \quad \textit{and} \quad R = \sum_{i=1}^{N} (\mathbb{B}^\circ \cap \mathcal{K}_i).$$

If $0 \notin \operatorname{cl}^*(B)$, then $B \subset R$, B is a base for $\mathcal{K}$, and $\mathcal{K}$ is pointed. Conversely, if $\mathcal{K} \neq \{0\}$ is pointed, and X^* is finite dimensional, then B is a base for $\mathcal{K}$.

Proof. Since the set $R = \sum_{i=1}^{N} (\mathbb{B}^\circ \cap \mathcal{K}_i)$ is a finite sum of weak* compact sets, it is weak* closed [3, page 25] and bounded, and hence weak* compact by Alaoglu's Theorem [13, Page 13]. In addition, this set is convex since it is the finite sum of convex sets. Hence $\operatorname{cl}^*(B) \subset R$ since $\operatorname{bdry}(\mathbb{B}^\circ) \cap \mathcal{K}_i \subset R$ for $i = 1, \dots, N$, and R is weak* closed and convex. Therefore, $\operatorname{cone}(B) \subset \operatorname{cone}(R) \subset \mathcal{K}$. We claim that $\mathcal{K} \subset \operatorname{cone}(B)$ which completes the proof that B is a base for $\mathcal{K}$. Indeed, given $x^* \in \mathcal{K}$, there exists $x_i^* \in \mathcal{K}_i,\ i = 1, \dots, N$ such that $x^* = x_1^* + \cdots + x_n^*$. If $x^* = 0$, then

$x^* \in \operatorname{cone}(B)$, so we may assume that at least one $x_i^* \neq 0$. Set $I = \left\{ i \,\middle|\, x_i^* \neq 0 \right\}$ and define $\lambda = \sum_{i \in I} \|x_i^*\|$, $w_i = x_i^* / \|x_i^*\|$, and $\mu_i = \lambda^{-1} \|x_i^*\|$ for $i \in I$. Then $x^* = \lambda \sum_{i \in I} \mu_i w_i \in \operatorname{cone}(B)$, so $\mathcal{K} \subset \operatorname{cone}(B)$. If $\mathcal{K}$ is not pointed, then there is a nonzero $x^* \in \mathcal{K} \cap (-\mathcal{K})$. So there exist positive λ_1 and λ_2 such that $\lambda_1 x^* \in B$ and $-\lambda_2 x^* \in B$, which implies $0 \in B$ by the convexity of B. The contradiction shows that $\mathcal{K}$ is pointed.

Conversely, if X^* is finite dimensional, then the compactness of B follows from that of the sets $\operatorname{bdry}(\mathbb{B}^\circ) \cap \mathcal{K}_i$. Hence B is not a base if and only if $0 \in B$, which is equivalent to $\mathcal{K}$ is not pointed. □

Proposition 2. *Let C, C_i, $i = 1, \ldots, N$ be as in Definition 1 and let $x \in C$.*

- *(i) If $D \subseteq C$ is such that each of the cones $K_i = \mathrm{cl}^*\left(\bigcup_{x \in D} N_{C_i}(x)\right)$, $i = 1, \ldots, N$, is convex and $0 \notin \mathrm{cl}^*(B_D)$ where $B_D = \operatorname{co}\left(\cup_{i=1}^N (\operatorname{bdry}(\mathbb{B}^\circ) \cap K_i)\right)$, then B_D is a base for the cone $K = \sum_{i=1}^N K_i$, for each $x \in D$ the set $B_x = \left(\cup_{i=1}^N (\operatorname{bdry}(\mathbb{B}^\circ) \cap N_{C_i}(x))\right)$ is a base for the cone $\sum_{i=1}^N N_{C_i}(x)$, and the inclusion (11) holds for all $x \in D$ uniformly in $\alpha = \inf_{z \in \mathrm{cl}^*(B_D)} \|z\| > 0$.*
- *(ii) If $D \subseteq C$ is such that the cone $\mathrm{cl}^*\left(\bigcup_{x \in D} N_C(x)\right)$ is convex and has a weak* sequentially compact base B, then there is an $\alpha > 0$ such that the inclusion (11) holds for all $x \in D$ uniformly in α.*
- *(iii) If the cone $(C^\infty)^\circ$ has a weak* sequentially compact base B where C^∞ denotes the recession cone of C, then there is an $\alpha > 0$ such that the inclusion (11) holds for all $x \in C$.*
- *(iv) If X is reflexive, then the hypothesis that the base B in Parts (ii) and (iii) is weak* sequentially compact can be replaced by the hypothesis that B is closed, and bounded.*

Proof. (i) The cones K_i, $i = 1, \ldots, N$ and the set B_D satisfy the hypotheses of Lemma 2, and so B_D is a base for K and $B_D \subset R_D = \sum_{i=1}^N \mathbb{B}^\circ \cap K_i$. Now since the set R_D is bounded, so is the set B_D. Hence the set $\mathrm{cl}^*(B_D)$ is weak* closed and bounded, and so is a convex weak* compact set [13, Page 13]. Since $0 \notin \mathrm{cl}^*(B_D)$, the origin can be properly separated from $\mathrm{cl}^*(B_D)$ and so $\inf_{z \in \mathrm{cl}^*(B_D)} \|z\| = \alpha > 0$. In particular, this implies that for each $x \in D$ we have

$$\inf_{z \in \mathrm{cl}^*(B_x)} \|z\| \geq \inf_{z \in \mathrm{cl}^*(B_D)} \|z\| = \alpha > 0\,.$$

Consequently, $0 \notin \mathrm{cl}^*(B_x)$ for every $x \in D$. Thus, setting $\mathcal{K}_i = N_{C_i}(x)$ and $B = B_x$ in Lemma 2 tells us that B_x is a base for the cone $\sum_{i=1}^N N_{C_i}(x)$ with $B_x \subset R_x = \sum_{i=1}^N (\mathbb{B}^\circ \cap N_{C_i}(x))$ for every $x \in D$.

Finally, given $x \in D$, Lemma [8, Lemma 3.4, Part 2] applies to yield the inclusion

$$\alpha\mathbb{B}^\circ \cap \operatorname{cone}(\operatorname{co}(0, B_x)) \subset \operatorname{co}(0, B_x)\,,$$

where $\operatorname{cone}(\operatorname{co}(0, B_x)) = \operatorname{cone}(B_x)$. Therefore,

$$\alpha\mathbb{B}^\circ \cap \left(\sum_{i=1}^N N_{C_i}(x)\right) \subset \alpha\mathbb{B}^\circ \cap \operatorname{cone}(B_x) \subset \operatorname{co}(0, B_x) \subset R_x. \tag{12}$$

That is, (11) holds at x.

(ii) The weak* closedness of B implies that B is a closed set in the norm topology, and so $\inf_{b\in B}\|b\| = \bar{\alpha} > 0$. Lemma [8, Lemma 3.4, Part 2] yields

$$\bar{\alpha}\mathbb{B}^\circ \cap \left(\sum_{i=1}^{N} N_{C_i}(x)\right) \subset \text{co}\left(0, \hat{B}_x\right),$$

where $\hat{B}_x = B \cap (\sum_{i=1}^{N} N_{C_i}(x))$. We claim that there is some $\beta > 0$ such that

$$\hat{B}_x \subset \sum_{i=1}^{N} \beta\mathbb{B}^\circ \cap N_{C_i}(x) \quad \forall x \in D. \tag{13}$$

Note that the claim (13) implies that (11) holds uniformly in x over D with $\alpha = \bar{\alpha}/\beta$.

Let us suppose that the claim (13) is not true. Then, with no loss in generality, there would exist sequences $\{x_n\} \subset D$, $\{z_n^*\} \subset B$ with $z_n^* \in \hat{B}_{x_n}$, $\{x_n^*\} \subset X^*$ with $x_n^* \in N_{C_1}(x_n)$ $n = 1, 2, \ldots$, and $\{y_n^*\} \subset X^*$ with $y_n^* \in \sum_{i=2}^{N} N_{C_i}(x_n)$ $n = 1, 2, \ldots$ such that $z_n^* = x_n^* + y_n^*$ with $\|x_n^*\| \geq n\ \forall n = 1, 2, \ldots$. Since $B \cap N_{C_1}(x_n)$ is a base for $N_{C_1}(x_n)$, there exists $t_n > 0$ such that $x_n^*/t_n \in B$ for all n. The weak* sequential compactness of B implies that B is bounded [36, Theorem 10, Page 125] and so $t_n \to \infty$. Since B is weak* sequentially compact, we can assume with no loss in generality that $\{x_n^*/t_n\}$ weak* converges to some $x^* \in B$. Since B is bounded, we also know that z_n^*/t_n converges strongly to zero.

Let $u \in \mathbb{B}$ such that $x^*(u) \geq 3/4\,\|x^*\|$. Then

$$\begin{aligned}
\|y_n^*\|/t_n &\geq |y_n^*(u)|/t_n = |z_n^*(u) - x_n^*(u)|/t_n \\
&\geq |x^*(u)| - |x^*(u) - x_n^*(u)/t_n| - \|z_n^*/t_n\| \\
&\geq \frac{1}{2}\|x^*\|
\end{aligned} \tag{14}$$

for all n sufficiently large. Since $B \cap (\sum_{i=2}^{N} N_{C_i}(x_n))$ is a base for $\sum_{i=2}^{N} N_{C_i}(x_n)$ and $y_n^*/t_n \in \sum_{i=2}^{N} N_{C_i}(x_n)$, there is some $\alpha_n > 0$ such that $\alpha_n y_n^*/t_n \in B$ for each n. This along with (14) yields $\alpha_n \leq 2M/\|x^*\|$ for n sufficiently large, where $M = \sup_{b\in B}\|b\|$. Hence $\{\alpha_n\}$ has a convergent subsequence. Without loss of generality, suppose that $\alpha_n \to \hat{\alpha} \geq 0$. Then, since

$$\alpha_n(z_n^*/t_n - x_n^*/t_n) = \alpha_n y_n^*/t_n \in B,$$

we have that $-\hat{\alpha}x^* \in B$, and so $0 \in B$ by the convexity of B. This contradicts the fact that B is a base, hence (13) must hold for some $\beta > 0$.

(iii) By inclusion (A.2) in Lemma A.1 of Appendix A,

$$\text{cl}^*\left(\bigcup_{x\in C} N_C(x)\right) \subset (C^\infty)^\circ,$$

with equality holding when X is reflexive. Since B is weak* sequentially compact and the set $\text{cl}^*\left(\bigcup_{x\in C} N_C(x)\right)$ is weak* closed, the set $\hat{B} = B \cap \text{cl}^*\left(\bigcup_{x\in C} N_C(x)\right)$ is necessarily a weak* sequentially compact base for the cone $\text{cl}^*\left(\bigcup_{x\in C} N_C(x)\right)$ that does not contain the origin. Therefore, the result follows from Part (ii).

(iv) By Alaoglu's Theorem [13, Page 13] and the Eberlein-Smulian Theorem [13, Page 18], every weak* closed and bounded subset of X^* is weak* sequentially compact whenever X is reflexive. □

By combining the results in Theorem 2 and Proposition 2 one can obtain a number of conditions that imply linear regularity. We provide a sample result of this type that is particularly simple to state.

Theorem 3. *Let $C,\ C_i,\ i = 1, \dots, N$ be as in Definition 1, let $x \in C$, and assume that X is reflexive. If strong CHIP (10) is satisfied at every point of C and the cone $(C^\infty)^\circ$ has a weak* closed and bounded base B, then the collection $\{ C_i \mid i = 1, \dots, N \}$ is linearly regular.*

Results for bounded linear regularity can similarly be obtained from the notion of *boundedly weak sharp minima*.

Definition 3. [8, Definition 6.1] *Let $S \subset X$ and let $f : X \mapsto \overline{\mathbb{R}}$ where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. The set $\tilde{S} := \arg\min \{ f(x) \mid x \in S \}$ is said to a set of boundedly weak sharp minima for f over the set S if for every $r > 0$ for which $\tilde{S} \cap r\mathbb{B} \neq \emptyset$ there is an $\alpha_r > 0$ such that*

$$f(x) \geq f(\bar{x}) + \alpha_r \operatorname{dist}(x \mid \tilde{S}) \tag{15}$$

for all $x \in S \cap r\mathbb{B}$, where $\bar{x}$ is any element of $\tilde{S}$.

The collection $\{ C_i \mid i = 1, \dots, N \}$ is boundedly linearly regular if and only if C is a set of bounded weak sharp minima for the function ρ, again, regardless of the choice of monotone norm. Therefore, we can apply [8, Theorem 6.3] to characterize bounded linear regularity.

Theorem 4. *If the collection $\{ C_i \mid i = 1, 2, \dots, N \}$ is boundedly linearly regular, then strong CHIP (10) holds at every point of C and for every $r > 0$ for which $C \cap r\mathbb{B} \neq \emptyset$ there is an $\alpha_r > 0$ such that*

$$\alpha_r \mathbb{B}^\circ \cap \left(\sum_{i=1}^{N} N_{C_i}(x) \right) \subset \sum_{i=1}^{N} [\mathbb{B}^\circ \cap N_{C_i}(x)] \quad \forall\, x \in C \cap r\mathbb{B}\,. \tag{16}$$

If it is assumed that X is either a Hilbert space or finite dimensional, then the converse is also true, that is, (16) and strong CHIP on C implies bounded linear regularity. Finally, if X is finite dimensional, then $\{ C_i \mid i = 1, 2, \dots, N \}$ is boundedly linearly regular if and only if strong CHIP holds on C and for every $\bar{x} \in C$ there exist $\epsilon > 0$ and $\alpha_{\bar{x}} > 0$ such that

$$\alpha_{\bar{x}} \mathbb{B}^\circ \cap \left(\sum_{i=1}^{N} N_{C_i}(x) \right) \subset \sum_{i=1}^{N} [\mathbb{B}^\circ \cap N_{C_i}(x)] \quad \forall\, x \in C \cap (\bar{x} + \epsilon\mathbb{B})\,. \tag{17}$$

Proof. Apply Theorem [8, Theorem 6.3] with $S = X$, $\tilde{S} = C$, and $f = \rho_1$, where ρ is defined in (7) with the norm taken to be the 1–norm. As in the proof of Theorem 2, we have $\partial\rho_1(x) = \sum_{i=1}^{N} \partial \operatorname{dist}(x \mid C_i)$.

The final statement of the Theorem follows from [8, Corollary 5.3]. □

As was the case for linear regularity, Propositions 1 and 2 can be applied to obtain sufficient conditions for bounded linear regularity. A sample result of this type is given below that uses Parts 3 and 4 of Proposition 2.

Theorem 5. *Suppose that X is either a Hilbert space or is finite dimensional and let $C,\ C_i\ i = 1, \ldots, N$ be as in Definition 1. If strong CHIP (10) is satisfied at every point of C and for every $r > 0$ for which $C \cap r\mathbb{B} \neq \emptyset$ the set $\mathrm{cl}^* \left(\bigcup_{x \in C \cap r\mathbb{B}} N_C(x)\right)$ is convex and has a weakly closed and bounded base B, then the collection $\{ C_i \mid i = 1, \ldots, N \}$ is bounded linearly regular.*

We now provide conditions under which bounded linear regularity implies linear regularity. We omit the proof of this result since it is an immediate consequence of [8, Theorem 6.5] and the elementary equivalence $\mathrm{dom}\ \left(\rho_1^*\right) = \sum_{i=1}^N [\mathbb{B}^\circ \cap \mathrm{bar}\ (C_i)]$.

Theorem 6. *Suppose that X is a reflexive Banach space. If C admits a decomposition of the form $C = K + D$, where K is a nonempty closed convex cone and D is a nonempty closed bounded convex set, then the collection $\{ C_i \mid i = 1, 2, \ldots, N \}$ is linearly regular if and only if it is boundedly linear regular. In addition, if X is assumed to be finite dimensional, then the decomposition $C = K + D$, holds if either (a) $0 \in \mathrm{ri}\ \left(\sum_{i=1}^N [\mathbb{B}^\circ \cap \mathrm{bar}\ (C_i)]\right)$, or (b) C is polyhedral.*

Remark 3. With some effort one can also establish Theorem 6 using the very recent result [27, Theorem 3.1] by choosing an appropriate function F. However, with no extra effort the result is an immediate consequence of the earlier result [8, Theorem 6.5].

Remark 4. The decomposition of a convex set into the sum of a bounded set and a cone has a long history in convex analysis. References to some of this history can be found in [8].

In finite dimensions, it is possible to establish the inclusion (11) or (16) under conditions weaker than those employed in Proposition 2.

Definition 4. *Let $\{ K_i \mid i = 1, \ldots, m \}$ be a collection of closed cones in the normed linear space X. We say that the collection $\{ K_i \mid i = 1, \ldots, m \}$ is additively regular if $\sum_{i=1}^m z_i = 0$ with $z_i \in K_i$ for $i = 1, \ldots, m$ implies that $z_i \in K_i \cap (-K_i)$ for $i = 1, \ldots, m$. The collection $\{ K_i \mid i = 1, \ldots, m \}$ is said to be strongly additively regular if $\sum_{i=1}^m z_i = 0$ with $z_i \in K_i$ for $i = 1, \ldots, m$ implies that $z_i = 0$ for $i = 1, \ldots, m$.*

If $\sum_{i=1}^m K_i$ is pointed, then the collection of $\{ K_i \mid i = 1, \ldots, m \}$ is additively regular. When X is a finite dimensional Hilbert space, we denote it by $\mathbb{R}^n$. We now give a dual characterization for additive regularity and consequences of this regularity property when $X = \mathbb{R}^n$. This result plays a crucial role in establishing the main result of this section.

Lemma 3. *Let $\{ K_i \mid i = 1, \ldots, m \}$ be a collection of closed cones of $\mathbb{R}^n$. Then the following is true.*

1. $\{ K_i \mid i = 1, \ldots, m \}$ *is additively regular if and only if* $\cap_{i=1}^m \mathrm{ri}\ K_i^o \neq \emptyset$.

2. *Suppose the collection* $\{ K_i \mid i = 1, \dots, m \}$ *is additively regular, and that* M *is a subspace. Then the collection* $\{ K_i \cap M \mid i = 1, \dots, m \}$ *is additively regular.*
3. *Let* $\{ I_i \mid i = 1, \dots, k \}$ *be a partition of the set* $\{1, \dots, m\}$. *Define* $K_{I_i} = \sum_{j \in I_i} K_j$. *Suppose that the collection* $\{ K_i \mid i = 1, \dots, m \}$ *is additively regular. Then for each* I_i, K_{I_i} *is a closed cone, and* $\left\{ K_{I_i} \mid i = 1, \dots, k \right\}$ *is additively regular.*

Proof. 1. Let $L \subset \prod_{i=1}^m \mathbb{R}^n = R^{n\times m}$ be given by $\{x = (x_1, \dots, x_m) \ : \ \sum_{i=1}^m x_i = 0\}$, where $x_i \in \mathbb{R}^n$ for $i = 1, \dots, m$. Then the collection $\{ K_i \mid i = 1, \dots, m \}$ is additively regular if and only if $L \cap \prod_{i=1}^m K_i$ is a subspace, which in turn is equivalent to the statement that $L^\perp + \prod_{i=1}^m K_i^o$ is a subspace, where $L^\perp = \{x = (x_1, \dots, x_m) \ : \ x_1 = \cdots = x_m\}$. But $L^\perp + \prod_{i=1}^m K_i^o$ is a cone. So $L^\perp + \prod_{i=1}^m K_i^o$ is a subspace if and only if $0 \in \text{ri}\, \left(L^\perp + \prod_{i=1}^m K_i^o\right)$. This means that $L^\perp \cap \text{ri}\, \left(\prod_{i=1}^m K_i^o\right) \neq \emptyset$, which is equivalent to $\cap_{i=1}^m \text{ri}\, \left(K_i^o\right) \neq \emptyset$.

Part 2 is evident, so we now prove Part 3. For each I_i the collection $\left\{ K_j \mid j \in I_i \right\}$ is additively regular, so the closedness of K_{I_i} follows from [32, Corollary 9.1.3]. The additive regularity of the collection $\left\{ K_{I_i} \mid i = 1, \dots, k \right\}$ follows from the additive regularity of the collection $\{ K_i \mid i = 1, \dots, m \}$. □

The main result of this section follows.

Theorem 7. *Suppose that* X *is a Hilbert space. Let* $\{ C_i \mid i = 1, \dots, N \}$ *be a collection of closed convex subsets of* X *with* $C = \bigcap_{i=1}^N C_i$ *nonempty. For statements 1,2, and 3, we assume that* $X = \mathbb{R}^n$.

1. *Let* $x \in C$. *If the collection* $\left\{ N_{C_i}(x) \mid i = 1, \dots, N \right\}$ *is additively regular, then strong CHIP (10) holds at* x *and (11) holds for some* $\alpha_x > 0$.
2. *Let* $r > 0$ *be such that* $C \cap r\mathbb{B} \neq \emptyset$. *If the collection* $\left\{ N_{C_i}(x) \mid i = 1, \dots, N \right\}$ *is additively regular at every point* x *in* $C \cap r\mathbb{B}$, *then the strong CHIP (10) holds at every point* x *in* $C \cap r\mathbb{B}$, *and there is an* $\alpha_r > 0$ *such that (16) holds. As a consequence, if the collection* $\left\{ N_{C_i}(x) \mid i = 1, \dots, N \right\}$ *is additively regular at every point of* C, *then the collection* $\{ C_i \mid i = 1, \dots, N \}$ *is boundedly linearly regular.*
3. *If* $\cap_{i=1}^N \text{ri}\, (C_i) \neq \emptyset$, *then the collection* $\left\{ N_{C_i}(x) \mid i = 1, \dots, N \right\}$ *is additively regular at every point of* C.
4. *If* $(\cap_{i=1}^{N-1} int(C_i)) \cap C_N \neq \emptyset$, *then the collection* $\left\{ N_{C_i}(x) \mid i = 1, \dots, N \right\}$ *is strongly additively regular at every point* x *of* C.

Proof. The proofs of Parts 1 and 2 follow the same pattern relying on a standard compactness argument. Therefore, we only provide the proof of Part 2.

2. We first show that strong CHIP (10) holds at every point of $C \cap r\mathbb{B}$. Let $x \in C \cap r\mathbb{B}$. By definition, $T_C(x) = \text{cl}\, (\text{cone}\, (C - x)) = \text{cl}\, \left(\text{cone}\, (\cap_{i=1}^N C_i - x)\right) = \text{cl}\, \left(\text{cone}\, (\cap_{i=1}^N (C_i - x))\right)$. By Part 1 of Lemma 3, $\cap_{i=1}^N \text{ri}\, \left(T_{C_i}(x)\right) \neq \emptyset$. It follows from [32, Theorem 6.5], that

$$\text{cl}\, \left(\cap_{i=1}^N \text{cone}\, (C_i - x)\right) = \cap_{i=1}^N \text{cl}\, (\text{cone}\, (C_i - x)) = \cap_{i=1}^N T_{C_i}(x).$$

So $T_C(x) = \cap_{i=1}^N T_{C_i}(x)$. By taking polars and by Part 3 of Lemma 3, we have the strong CHIP (10) at x: $N_C(x) = \text{cl}\, \left(\sum_{i=1}^N N_{C_i}(x)\right) = \sum_{i=1}^N N_{C_i}(x)$.

We next show that there is an α_r such that (16) holds. We only do this for $N = 2$. The general case follows easily by the principle of mathematical induction on N. For $i = 1, 2$, let E_i be a subspace orthogonal to the affine hull of C_i. Then$N_{C_i}(x) \cap (-N_{C_i}(x)) = E_i$ for all $x \in C_i$. Set $E = E_1 \cap E_2$. Then $E \subset N_{C_i}(x)$ for $i = 1, 2$ and $C_i \subset E^{\perp} + x$ for $i = 1, 2$ and $x \in C$. It follows that $(C_1 + E) \cap (C_2 + E) = C + E$, and $N_{C_i+E}(x) = N_{C_i}(x) \cap E^{\perp}$ for $i = 1, 2$ and $x \in C$. Since $\{N_{C_1}(x), N_{C_2}(x)\}$ is additively regular for all $x \in C \cap r\mathbb{B}$, by Part 2 of Lemma 3, $\{N_{C_1+E}(x), N_{C_2+E}(x)\}$ is additively regular for all $x \in C \cap r\mathbb{B}$. This implies that, for each $x \in C \cap r\mathbb{B}$,

$$\begin{aligned}(N_{C_1}(x) + N_{C_2}(x)) \cap E^{\perp} &= N_C(x) \cap E^{\perp} = N_{C+E}(x) = N_{(C_1+E)\cap(C_2+E)}(x) \\ &= N_{C_1+E}(x) + N_{C_2+E}(x) \\ &= N_{C_1}(x) \cap E^{\perp} + N_{C_2}(x) \cap E^{\perp}.\end{aligned}$$

We now show that, for the collection $\{C_1 + E, C_2 + E\}$, there is a positive $\bar{\alpha}_r$

$$\bar{\alpha}_r \mathbb{B}^{\circ} \cap \left(\sum_{i=1}^{2} N_{C_i+E}(x)\right) \subset \sum_{i=1}^{2} [\mathbb{B}^{\circ} \cap N_{C_i+E}(x)] \quad \forall x \in C \cap r\mathbb{B}\,. \tag{18}$$

Assume the result is false. Then, there must exist sequences $\{x^j\} \subset C \cap r\mathbb{B}$, $\bar{\alpha}_j \searrow 0$, and $z_i^j \in N_{C_i}\left(x^j\right) \cap E^{\perp}$ with $i = 1, 2$, $\left\|z_1^j + z_2^j\right\| = 1$ and $j = 1, 2, \ldots$, such that $\sum_{i=1}^{2} z_i^j \notin \sum_{i=1}^{2} \left[\bar{\alpha}_j^{-1} \mathbb{B}^{\circ} \cap N_{C_i+E}\left(x^j\right)\right]$. Consequently, $\bar{\alpha}_j^{-1} < \max_{1 \le i \le 2}\{\left\|z_i^j\right\|\}$. Due to the compactness of the set $C \cap r\mathbb{B}$ and the finiteness of the index set $\{1, 2\}$, we may assume with no loss in generality that there exist $\bar{x} \in C \cap r\mathbb{B}$ and $\bar{z}_i \in N_{C_i+E}(\bar{x})$ for $i = 1, 2$ with $\|\bar{z}_i\| = 1$ and $i = 1, 2$ such that $x^j \to \bar{x}$ and $z_i^j / \left\|z_1^j\right\| \to \bar{z}_i \neq 0$. But then $\bar{z}_1 \in E_1 \cap E_2 \cap E^{\perp} = \{0\}$. The contradiction shows that (18) holds for the collection $\{C_1 + E, C_2 + E\}$.

For any $x \in C \cap r\mathbb{B}$, and $z \in \mathbb{B}^{\circ} \cap N_C(x)$, by $N_C(x) = N_{C_1}(x) \cap E^{\perp} + N_{C_2}(x) \cap E^{\perp} + E$, we have

$$\begin{aligned} z &\in \mathbb{B}^{\circ} \cap (N_{C_1}(x) \cap E^{\perp} + N_{C_2}(x) \cap E^{\perp}) + \mathbb{B}^{\circ} \cap E \\ &\subset \sum_{i=1}^{2} (\beta_r \mathbb{B}^{\circ} \cap N_{C_i+E}(x)) + \mathbb{B}^{\circ} \cap E \quad \text{where } \beta_r = \bar{\alpha}_r^{-1} \\ &\subset \sum_{i=1}^{2} (\beta_r + 1)\mathbb{B}^{\circ} \cap (N_{C_i+E}(x) + E) \\ &\subset \sum_{i=1}^{2} (\beta_r + 1)\mathbb{B}^{\circ} \cap N_{C_i}(x)\,. \end{aligned}$$

Setting $\alpha_r = (\beta_r + 1)^{-1}$ establishes (16) for the collection $\{C_1, C_2\}$.

To see the final statement of Part 2, we observe that the first statement establishes that for every $r > 0$ there is an $\alpha_r > 0$ such that (16) holds, and that the strong CHIP (10) holds at every point of C. Therefore, we obtain the result from Theorem 4.

3. For any point $x \in C$, we have

$$\cap_{i=1}^{N} \text{ri } \left(T_{C_i}(x)\right) = \cap_{i=1}^{N} \text{ri } (\text{cone}\,(C_i - x)) = \cap_{i=1}^{N} \text{cone}\,(\text{ri } (C_i - x)) \supset \cap_{i=1}^{N} \text{ri } (C_i - x) = \cap_{i=1}^{N} (\text{ri } (C_i) - x) \neq \emptyset,$$

where the second equality follows from a remark after [32, Corollary 6.8.1]. By Part 1 of Lemma 3, the collection $\left\{ N_{C_i}(x) \mid i = 1, \ldots, N \right\}$ is additively regular.

4. If the result were false, there would exist $\bar{x} \in C$ such that $\bar{z}^i \in N_{C_i}(\bar{x})$, $i = 1, \ldots, N$ with $\sum_{i=1}^{N} \bar{z}^i = 0$ but some $\bar{z}^{i_0} \neq 0$. With no loss in generality, $1 \leq i_0 \leq N-1$ since if $i_0 = N$ the condition $\sum_{i=1}^{N} \bar{z}^i = 0$ implies the existence of a j_0 between 1 and $N-1$ with $\bar{z}^{j_0} \neq 0$. In addition, we may assume that $\left\|\bar{z}^{i_0}\right\| = 1$. Set $\hat{C}_{i_0} = \left(\bigcap_{\substack{i=1\\ i\neq i_0}}^{N-1} C_i\right) \cap C_N$, and let $\hat{x} \in \left(\bigcap_{i=1}^{N-1} \text{int } (C_i)\right) \cap C_N$. By [33, Theorem 20] and the condition $\sum_{i=1}^{N} \bar{z}^i = 0$, we have $-\bar{z}^{i_0} \in \sum_{\substack{i=1\\ i\neq i_0}}^{N} N_{C_i}(\bar{x}) = N_{\hat{C}_{i_0}}(\bar{x})$. Since $\hat{x} \in \text{int }\left(C_{i_0}\right)$, there exists $\delta > 0$ such that $\hat{x} + \delta \bar{z}^{i_0} \in \text{int }\left(C_{i_0}\right)$. By combining these facts we have that, $-\bar{z}^{i_0} \in N_{\hat{C}_{i_0}}(\bar{x})$ with $\hat{x} - \bar{x} \in T_{\hat{C}_{i_0}}(\bar{x})$, and $\bar{z}^{i_0} \in N_{C_{i_0}}(\bar{x})$ with $\hat{x} + \delta \bar{z}^{i_0} - \bar{x} \in T_{C_{i_0}}(\bar{x})$. Therefore,

$$\delta = \delta \left\|\bar{z}^{i_0}\right\|_2^2 = \left\langle \bar{z}^{i_0}, \delta \bar{z}^{i_0} \right\rangle \leq \left\langle \bar{z}^{i_0}, \hat{x} + \delta \bar{z}^{i_0} - \bar{x} \right\rangle \leq 0.$$

This contradiction establishes the validity of the statement in Part 4. □

Theorem 7 can be used to provide an alternative proof of Bauschke's Theorem [4, Theorem 5.6.2] on bounded linear regularity in finite dimensions.

4. Error bounds for convex inequalities

In this section we apply our results on weak sharp minima to derive necessary and sufficient conditions under which a global error bound exists for the convex inequality system

$$h(x) \leq 0 \quad x \in C, \tag{19}$$

where it is assumed that $h\colon X \mapsto \overline{\mathbb{R}}$ is lower semi-continuous, convex, and not identically $+\infty$, and the set $C \subset X$ is nonempty, closed, and convex. By a global error bound for (19), we mean the existence of a constant $\alpha > 0$ such that

$$\alpha \text{dist}\,(x \mid \Sigma) \leq \text{dist}\,(x \mid C) + h_+(x) \quad \forall\, x \in X, \tag{20}$$

where

$$\Sigma = \{ x \mid x \in C,\ h(x) \leq 0 \}.$$

In the case where the function h is linear such global error bounds are called *Hoffman* bounds [18]. Our analysis is based on the observation that the condition (20) is equivalent to the statement that the function $f\colon X \mapsto \overline{\mathbb{R}}$, given by

$$f(x) = \text{dist}\,(x \mid C) + h_+(x), \tag{21}$$

has Σ as a set of weak sharp minima.

Conditions for the existence of a global error bound for the system (19) can be used to derive similar conditions for more general convex inequality systems of the form

$$h_t(x) \le 0, \quad t \in T, \quad \text{and} \quad x \in C \tag{22}$$

where T is an index set (possibly infinite), for each $t \in T$ the functions $h_t : X \mapsto \overline{\mathbb{R}}$ $i = 1, \dots, N$ are lower semi-continuous, convex, and not identically $+\infty$, and the set $C \subset X$ is nonempty, closed, and convex. There are many ways in which this can be accomplished [10, 23, 40, 39]. But perhaps the simplest is to define $h(x) = \sup\{\, h_t(x) \mid t \in T \,\}$ for each $x \in X$. Then h is convex and further conditions can be imposed on the domains of the functions h_t to guarantee that the function h satisfies the conditions stated in (19). We provide a few sample results of this type at the end of this section.

Error bounds for the system (19) have been studied by many authors [2, 12, 20–23, 40, 39]. Our development is primarily motivated by the finite dimensional results obtained by Lewis and Pang in [22]. However, we will also see close connections to the very recent results appearing in [23, 40, 39]. Lewis and Pang note the connection to weak sharp minima in [22] but do not pursue it. We show that very general error bound results are easily obtained from the more general results for weak sharp minima. Indeed, using the results from Part I [8] we obtain a richer variety of results while recovering those that appear in [22] under weaker hypotheses. An immediate consequence of [8, Theorem 2.3] is the following characterizations of the existence of the global error bound (20). No proofs are needed.

Theorem 8. *Let $h : X \mapsto \overline{\mathbb{R}}$ and C be as given in (19).*
Let $\alpha > 0$ and consider the following statements:

1. *The global error bound (20) holds.*
2. *For all $x \in \Sigma$, $\alpha\mathbb{B}^\circ \cap N_\Sigma(x) \subset \mathbb{B}^\circ \cap N_C(x) + \partial h_+(x)$.*
3. *For all $x \in \Sigma$ and $d \in T_{\mathrm{dom}\,(h)}(x)$, $\alpha \operatorname{dist}(d \mid T_\Sigma(x)) \le \operatorname{dist}(d \mid T_C(x)) + h'_+(x; d)$.*
4. $\alpha\mathbb{B}^\circ \cap \left[\bigcup_{x\in\Sigma} N_\Sigma(x)\right] \subset \bigcup_{x\in\Sigma}\left[\mathbb{B}^\circ \cap N_C(x) + \partial h_+(x)\right]$.
5. *For all $x \in \Sigma$ and $d \in T_{\mathrm{dom}\,(h)}(x) \cap N_\Sigma(x)$, $\alpha\,\|d\| \le \operatorname{dist}(d \mid T_C(x)) + h'_+(x; d)$.*
6. *The inclusion $\hat\alpha\mathbb{B}^\circ \subset \mathbb{B}^\circ \cap N_C(x) + \partial h_+(x) + \left[T_{\mathrm{dom}\,(h)}(x) \cap N_\Sigma(x)\right]^\circ$ holds for all $0 \le \hat\alpha < \alpha$ and $x \in \Sigma$.*
7. *For all $y \in \mathrm{dom}\,(h)$, $\alpha \operatorname{dist}(y \mid \Sigma) \le \operatorname{dist}(y - p \mid T_C(p)) + h'_+(p; y - p)$ where $p = P(y \mid \Sigma)$.*

Statements 1 through 4 are equivalent. If, in addition, X is assumed to be a Hilbert space, then these statements are equivalent to each of the statements 5, 6, and 7.

Remark 5. The equivalence of statements 2. and 3. in Theorem 8 is given in [40, Theorem 2.3]. However, the authors of [40] were unaware of the results in [8] and so needed to reconstruct some of the machinery in [8] to establish this result.

In order to understand Theorem 8 better, one needs a better description of the subdifferential of $h_+(x)$. Although this is a simple max-function there is a bit of a twist in that it may be the case that $\{x \mid h(x) < 0\} \not\subset \operatorname{int}(\operatorname{dom}(h))$. When this occurs, the standard formula for the subdifferential of max-functions does not apply.

Proposition 3. *Let* $h\colon X \mapsto \overline{\mathbb{R}}$ *be as given in (19).*

(i) If $\operatorname{dom}(h) \cap \{x \mid h(x) > 0\} \neq \emptyset$, *then*

$$\partial h_+(x) = \begin{cases} \partial h(x) & , \text{ if } h(x) > 0, \\ N_{\operatorname{dom}(h)}(x) & , \text{ if } h(x) < 0, \text{ and} \\ \operatorname{cl}^*\left(N_{\operatorname{dom}(h)}(x) \cup \left(\bigcup_{\lambda\in[0,1]} \lambda\partial h(x)\right)\right) & , \text{ if } h(x) = 0. \end{cases} \tag{23}$$

(ii) If $\operatorname{dom}(h) \cap \{x \mid h(x) > 0\} = \emptyset$, *then* $h_+ = \psi_{\operatorname{lev}_h(0)} = \psi_{dom(h)}$ *and* $\partial h_+(x) = N_{\operatorname{lev}_h(0)}(x) = N_{dom(h)}(x)$ *for all* $x \in \operatorname{dom}(h)$.
(iii) If $\bar{x} \in C \cap \{x \mid h(x) < 0\}$,

$$\partial h_+(\bar{x}) = N_{\operatorname{dom}(h)}(\bar{x}) = N_{\operatorname{lev}_h(0)}(\bar{x}), \text{ and } N_\Sigma(\bar{x}) = N_{C\cap\operatorname{dom}(h)}(\bar{x}) .$$

Remark 6. The expression for $\partial h_+(x)$ in (23) for the case $h(x) = 0$ follows from a more general result of Volle [35, Theorem 2].

Remark 7. Note that if $x \in h^{-1}(0)$ and $\partial h(x) = \emptyset$, then $\partial h_+(x) = N_{\operatorname{dom}(h)}(x)$. On the other hand, if $x \in h^{-1}(0)$ and $\partial h(x) \neq \emptyset$, then $\partial h_+(x) = \bigcup_{\lambda\in[0,1]} \lambda\partial h(x)$ since in this case $N_{\operatorname{dom}(h)}(x) \subset \partial h(x)$.

Proof. Part (ii) is a straightforward consequence of the definitions and so we only prove (i). Define $g_i\colon X \mapsto \overline{\mathbb{R}}$ for $i = 1, 2$ by $g_1(x) \equiv 0$ and $g_2(x) = h(x)$, and set $I(x) = \{i \mid g_i(x) = h_+(x)\}$. The inclusions $\partial h_+(x) \supseteq \bigcup_{i\in I(x)} \partial g_i(x)$ and $\partial h_+(x) \supseteq N_{\operatorname{dom}(h)}(x)$ are well-known and easily verified. These two inclusions immediately imply that the right hand side of (23) is contained in the left hand side of (23) since the set $\partial h_+(x)$ is weak* closed and convex. Therefore, we need only prove the reverse inclusion. We consider 3 cases: $h(x) > 0$, $0 > h(x)$, and $h(x) = 0$.

If $h(x) > 0$, then $h_+(y) = h(y)$ on a neighborhood of x due to the lower semi-continuity of h. Consequently, $\partial h_+(y) = \partial h(y)$ for all y near x.

If $0 > h(x)$, then, for all $d \in X$ for which there is a $\bar{t} > 0$ such that $x + td \in \operatorname{dom}(h)$ for all $t \in [0, \bar{t}]$, we have $h(x + \lambda\bar{t}d) \leq h(x) + \lambda[h(x + \bar{t}d) - h(x)]$ for $0 < \lambda \leq 1$, and so for $t > 0$ sufficiently small we have $h(x + td) < 0$. Consequently, for such directions d we have $h'_+(x; d) = 0$. On the other hand, if $x + td \notin \operatorname{dom}(h)$ for all small $t > 0$, then $h'_+(x; d) = +\infty$. Therefore, by [33, Theorem 11], $\partial h_+(x) = N_{\operatorname{dom}(h)}(x)$.

If $h(x) = 0$, then result is an immediate consequence of [35, Theorem 2].

We now prove (iii). The equivalence $\partial h_+(\bar{x}) = N_{\operatorname{dom}(h)}(\bar{x})$ has already been established in (23). The second equation in (iii) is equivalent to the equation $T_{\operatorname{lev}_h(0)}(\bar{x}) = T_{\operatorname{dom}(h)}(\bar{x})$. Clearly, $T_{\operatorname{lev}_h(0)}(\bar{x}) \subset T_{\operatorname{dom}(h)}(\bar{x})$, and so we need to show the reverse inclusion. For this let $d \in X$ be such that there is a $\bar{t} > 0$ for which $\bar{x} + td \in \operatorname{dom}(h)$ for all $t \in [0, \bar{t}]$. Then, $h(\bar{x} + \lambda\bar{t}d) \leq h(\bar{x}) + \lambda[h(\bar{x} + \bar{t}d) - h(\bar{x})]$ for $0 < \lambda \leq 1$, so that $h(\bar{x} + td) < 0$ for all $t > 0$ sufficiently small. Consequently, $T_{\operatorname{dom}(h)}(\bar{x}) \subset T_{\operatorname{lev}_h(0)}(\bar{x})$.

The final equation in (iii) follows in a manner similar to that of the second equation. We show the equivalent formula $T_\Sigma(\bar{x}) = T_{C\cap\operatorname{dom}(h)}(\bar{x})$. Since $\Sigma \subset C \cap \operatorname{dom}(h)$, we need only show that $T_{C\cap\operatorname{dom}(h)}(\bar{x}) \subset T_\Sigma(\bar{x})$. For this let $d \in X$ be such that there is a $\bar{t} > 0$ for which $\bar{x} + td \in C \cap \operatorname{dom}(h)$ for all $t \in [0, \bar{t}]$. Then, $h(\bar{x} + \lambda\bar{t}d) \leq$

$h(\bar{x}) + \lambda[h(\bar{x}) - h(\bar{x} + \bar{t}d)]$, so that $h(\bar{x} + td) < 0$ and $\bar{x} + td \in C$ for all $t > 0$ sufficiently small. Consequently, $T_{C\cap\mathrm{dom}\,(h)}(\bar{x}) \subset T_\Sigma(\bar{x})$. □

There are a number of conditions under which the characterizations given in Theorem 8 can be refined. We follow the pattern of results given in [22] and consider conditions under which the solution set Σ can be replaced by the set $C \cap f^{-1}(0)$.

Theorem 9. *Let the hypotheses of Theorem 8 hold and let $\hat{\alpha} > 0$. Consider the following statements:*

2'. For all $x \in C \cap h^{-1}(0)$, $\hat{\alpha}\mathbb{B}^\circ \cap N_\Sigma(x) \subset \mathbb{B}^\circ \cap N_C(x) + \partial h_+(x)$.
3'. For all $x \in C \cap h^{-1}(0)$ and $d \in T_{\mathrm{dom}\,(h)}(x)$,
$\hat{\alpha}\mathrm{dist}\,(d \mid T_\Sigma(x)) \le \mathrm{dist}\,(d \mid T_C(x)) + h'_+(x; d)$.
4'. $\hat{\alpha}\mathbb{B}^\circ \cap \left[\bigcup_{x\in C\cap h^{-1}(0)} N_\Sigma(x)\right] \subset \bigcup_{x\in C\cap h^{-1}(0)} \left[\mathbb{B}^\circ \cap N_C(x) + \partial h_+(x)\right]$.
5'. For all $x \in C \cap h^{-1}(0)$ and $d \in T_{\mathrm{dom}\,(h)}(x) \cap N_\Sigma(x)$,
$\hat{\alpha}\,\|d\| \le \mathrm{dist}\,(d \mid T_C(x)) + h'_+(x; d)$.
6'. The inclusion $\tilde{\alpha}\mathbb{B}^\circ \subset \mathbb{B}^\circ \cap N_C(x) + \partial h_+(x) + \left[T_{\mathrm{dom}\,(h)}(x) \cap N_\Sigma(x)\right]^\circ$ holds for all $0 \le \tilde{\alpha} < \alpha$ and $x \in C \cap h^{-1}(0)$.

In addition, consider the following hypotheses:

(a) $\{x \in C \mid h(x) < 0\} \subset \mathrm{int}\,(\mathrm{dom}\,(h))$.
(b) The pair of sets $\{C, \mathrm{lev}_h\,(0)\}$ is linearly regular.
(c) The pair of sets $\{C, \mathrm{dom}\,(h)\}$ satisfies (9) for all $x \in C \cap \mathrm{dom}\,(h)$.
(d) X is a Hilbert space and $P(\mathrm{dom}\,(h) \mid C) \subset \mathrm{dom}\,(h)$.

If (a), (b), or (c) holds, then each of the conditions 2', 3', and 4' imply that the global error bound (20) holds for some $\alpha \in (0, \hat{\alpha}]$ ($\alpha = \min\{\hat{\alpha}, 1\}$ under (a)). If, in addition, X is a Hilbert space, then each of the conditions 2'– 6' imply that the global error bound (20) holds for some $\alpha \in (0, \hat{\alpha}]$ ($\alpha = \min\{\hat{\alpha}, 1\}$ under (a)). Finally, condition (d) implies condition (c).

Remark 8. When X is assumed to be finite dimensional, hypotheses (a) and (d) correspond to conditions (8) and (9) in Lewis and Pang [22], respectively. Therefore, Part 5' recovers Theorem 2 in [22]

Remark 9. Parts 2', 3', 4', and 6' as well as conditions (b) and (c) have not previously appeared in the literature. When dom (h) is closed, by Theorem 2, condition (c) is equivalent to the linear regularity of the pair $\{C, \mathrm{dom}\,(h)\}$.

Proof. If $\Sigma = C \cap h^{-1}((-\infty, 0]) = C \cap h^{-1}(0)$, the result follows from Theorem 8. Hence we assume for the remainder of the proof that the set $\{x \in C \mid h(x) < 0\} = C \cap h^{-1}((-\infty, 0)) \ne \emptyset$.

The proof of the equivalence of the conditions 2, 3, 4, 5, and 6 in [8, Theorem 2.3] is pointwise with respect to elements of $\tilde{S}$. Therefore, since Theorem 8 shows that the conditions in [8, Theorem 2.3] are equivalent to the corresponding conditions in Theorem 8 with $f(x) = \mathrm{dist}\,(x \mid C) + h_+(x)$, $\tilde{S} = \Sigma$, and $S = \mathrm{dom}\,(f) = \mathrm{dom}\,(h)$, the conditions 2'– 4' are equivalent in the general case, and are equivalent to the conditions 5' and 6' in the Hilbert space setting. Hence, to establish the result, we need only show that one of the conditions 2'– 4' implies the corresponding condition in Theorem 8 for some $\alpha \in (0, \hat{\alpha}]$ (with $\alpha = \min\{\hat{\alpha}, 1\}$ if (a)). To this end, we show that 3' implies 3 in Theorem 8 when it is assumed that (a) holds, and show that 2' implies 2 Theorem 8 when (b) holds. This is done by showing that there is an $\bar{\alpha} > 0$ such that 2' or 3' is automatically satisfied on the set $\hat{\Sigma} = C \cap \{x \mid h(x) < 0\}$, whereby the corresponding result, 2 or 3 respectively, is satisfied with $\alpha = \min\{\bar{\alpha}, \hat{\alpha}\}$ since $\Sigma = \hat{\Sigma} \cup (C \cap h^{-1}(0))$.

If (a) holds, then, in fact, $\epsilon > 0$ may be chosen so that $h(x) < 0$ for all $x \in \bar{x} + \epsilon\mathbb{B}$ and $\Sigma \cap (\bar{x} + \epsilon\mathbb{B}) = C \cap (\bar{x} + \epsilon\mathbb{B})$. Therefore, $h'_+(\bar{x}; d) = 0$ for all $d \in X$ and $N_\Sigma(\bar{x}) = N_C(\bar{x})$, and so, by [8, Theorem A.1, Part 6], $\operatorname{dist}(d \mid T_\Sigma(\bar{x})) = \operatorname{dist}(d \mid T_C(\bar{x})) + h'_+(\bar{x}; d)$ for all $d \in T_{\operatorname{dom}(h)}(\bar{x})$. Consequently, 3' implies 3 with $\alpha = \min\{\hat{\alpha}, 1\}$.

If (b) holds, then Theorem 2 shows that there exists $\bar{\alpha} > 0$ such that

$$\bar{\alpha}\mathbb{B}^\circ \cap N_\Sigma(x) \subset \mathbb{B}^\circ \cap N_C(x) + \mathbb{B}^\circ \cap N_{\operatorname{lev}_h(0)}(x) \tag{24}$$

for all $x \in \Sigma$. Let $\bar{x} \in C \cap \{x \mid h(x) < 0\}$. By combining (24) with Lemma 3(iii) we obtain

$$\bar{\alpha}\mathbb{B}^\circ \cap N_\Sigma(\bar{x}) \subset \mathbb{B}^\circ \cap N_C(\bar{x}) + N_{\operatorname{dom}(h)}(\bar{x}) = \mathbb{B}^\circ \cap N_C(\bar{x}) + \partial h_+(\bar{x}). \tag{25}$$

Hence, 2' implies 2 with $\alpha = \min\{\hat{\alpha}, \bar{\alpha}\}$.

If (c) holds, then one establishes (25) as when (b) holds from which the result follows.

Suppose (d) holds and set $\hat{C} = C \cap \operatorname{dom}(h)$. Then $P(x \mid C) = P(x \mid \hat{C})$ for all $x \in \operatorname{dom}(h)$, or equivalently, $\operatorname{dist}(x \mid C) + \psi_{\operatorname{dom}(h)}(x) = \operatorname{dist}(x \mid \hat{C}) + \psi_{\operatorname{dom}(h)}(x) \quad \forall x \in X$. Therefore, [33, Theorem 20] and [8, Theorem A.1, Part 5] imply that $\mathbb{B}^\circ \cap N_{\hat{C}}(x) \subset \mathbb{B}^\circ \cap N_{\hat{C}}(x) + N_{\operatorname{dom}(h)}(x) = \mathbb{B}^\circ \cap N_C(x) + N_{\operatorname{dom}(h)}(x)$ for all $x \in \hat{C}$. Hence, for any $x \in \hat{C}$ and $z \in \mathbb{B}^\circ \cap N_{\hat{C}}(x)$ there exist $z^1 \in \mathbb{B}^\circ \cap N_C(x)$ and $z^2 \in N_{\operatorname{dom}(h)}(x)$ with $z = z^1 + z^2$, and so $\|z^2\| \le 2$. Consequently, $\frac{1}{2}\mathbb{B}^\circ \cap N_{\hat{C}}(x) \subset \mathbb{B}^\circ \cap N_C(x) + \mathbb{B}^\circ \cap N_{\operatorname{dom}(h)}(x)\, \forall x \in \hat{C}$. Therefore, the pair of sets $\{C, \operatorname{dom}(h)\}$ satisfies (9) for all $x \in \hat{C}$. □

We now turn to the study of sufficient conditions for the existence of the error bound (20). The two key conditions in our study are the Slater condition, $\{x \mid h(x) < 0\} \neq \emptyset$, and the linear regularity of the pair of sets $\{C, \operatorname{lev}_h(0)\}$. The linear regularity condition already appears in the previous result and is a corner stone of the analysis given by Bauschke, Borwein, and Li as well [5]. It is easy to see that the global error bound (20) implies the linear regularity of the pair $\{C, \operatorname{lev}_h(0)\}$.

Theorem 10. *Let h and C be as given in (19). If the global error bound (20) holds, then the pair* $\{C, \operatorname{lev}_h(0)\}$ *is linearly regular.*

Proof. By statement 2 in Theorem 8, $\alpha\mathbb{B}^\circ \cap N_\Sigma(x) \subset \mathbb{B}^\circ \cap N_C(x) + \partial h_+(x)$ for all $x \in \Sigma$. Since every lower semi-continuous convex function $f : X \mapsto \overline{\mathbb{R}}$ satisfies the inclusion $\operatorname{cone}(\partial f(x)) \subset N_{\operatorname{lev}_f(f(x))}(x)$ at points in $x \in \operatorname{dom}(f)$, this implies that $\alpha\mathbb{B}^\circ \cap N_\Sigma(x) \subset \mathbb{B}^\circ \cap N_C(x) + N_{\operatorname{lev}_h(0)}(x) \quad \forall x \in \Sigma$. Therefore, given $x \in \Sigma$ and $z \in \alpha\mathbb{B}^\circ \cap N_\Sigma(x)$ there exist $z^1 \in \mathbb{B}^\circ \cap N_C(x)$ and $z^2 \in N_{\operatorname{lev}_h(0)}(x)$ with $z = z^1 + z^2$. In particular, this implies that $\|z^2\| \le \alpha + 1$. It follows that $\frac{\alpha}{\alpha+1}\mathbb{B}^\circ \cap N_\Sigma(x) \subset \mathbb{B}^\circ \cap N_C(x) + \mathbb{B}^\circ \cap N_{\operatorname{lev}_h(0)}(x)$ for all $x \in \Sigma$, and so the result follows from Theorem 2. □

Our next result comes from convex analysis folklore. Its proof parallels the finite dimensional proof given by Rockafellar in [32, Theorem 23.7]. This basic result shows that the Slater condition implies that the Abadie constraint qualification,

$$\text{(ACQ)} \qquad N_{\mathrm{lev}_h(0)}(x) = \mathrm{cl}^*(\mathrm{cone}(\partial h(x))) \quad \forall x \in h^{-1}(0) \cap \mathrm{dom}\,(\partial h),$$

is satisfied. We begin with the following technical lemma based on [32, Theorem 7.6].

Lemma 4. *Let $g\colon X \mapsto \overline{\mathbb{R}}$ be a convex function on the normed linear space X, and suppose that $\mu \in \mathbb{R}$ is such that the set $\{x \mid g(x) < \mu\}$ is nonempty. Then* $\mathrm{cl}\,(\{x \mid g(x) < \mu\}) = \{x \mid (\mathrm{cl}\,g)(x) \le \mu\}$.

Proof. Since $\mathrm{cl}\,g$ is the lower semi-continuous hull of g, the set $\{x \mid (\mathrm{cl}\,g)(x) \le \mu\}$ is closed and

$$(\mathrm{cl}\,g)(x) \le g(x) \quad \forall\, x \in X. \tag{26}$$

Now if $\bar{x} \in \mathrm{cl}\,(\{x \mid g(x) < \mu\})$, then there is a sequence $\{x^i\} \in \{x \mid g(x) < \mu\}$ with $x^i \to \bar{x}$. Hence, by (26), $(\mathrm{cl}\,g)(x^i) \le g(x^i) < \mu$ for all $i = 1, 2, \ldots$. Since $\mathrm{cl}\,g$ is lower semi-continuous, $(\mathrm{cl}\,g)(\bar{x}) \le \mu$ and so $\bar{x} \in \{x \mid (\mathrm{cl}\,g)(x) \le \mu\}$. Thus, $\mathrm{cl}\,(\{x \mid g(x) < \mu\})$ is a closed subset of the set $\{x \mid (\mathrm{cl}\,g)(x) \le \mu\}$.

We now show the reverse inclusion. Let $\bar{x} \in \{x \mid (\mathrm{cl}\,g)(x) \le \mu\}$, let $\hat{x} \in \{x \mid g(x) < \mu\}$, and set $0 < \beta = \mu - g(\hat{x})$. Since $\mathrm{cl}\,(\mathrm{epi}\,(g)) = \mathrm{epi}\,(\mathrm{cl}\,(g))$, there exists a sequence $\{(x^i, \mu_i)\} \subset \mathrm{epi}\,(g)$ with $\mu_i \to (\mathrm{cl}\,g)(\bar{x}) \le \mu$ and $x^i \to \bar{x}$ If the sequence $\{\mu_i\}$ contains a subsequence $\{\mu_i\}_J$ such that $\mu_i < \mu$ for all $i \in J$, then $\bar{x} \in \mathrm{cl}\,(\{x \mid g(x) < \mu\})$. Hence, with no loss in generality, we may assume that $0 \le \delta_i = \mu_i - \mu$ for all $i = 1, 2, \ldots$. Let $\{\epsilon_i\} \subset (0, \frac{\beta}{2})$ satisfy $\epsilon_i \downarrow 0$ and set $\lambda_i = \frac{\beta - \epsilon_i}{\beta + \delta_i}$ so that $0 < \frac{\beta}{2(\beta + \delta_i)} \le \lambda_i < 1$ for all $i = 1, 2, \ldots$. Observe that

$$\begin{aligned} g((1-\lambda_i)\hat{x} + \lambda_i x^i) &\le (1-\lambda_i) g(\hat{x}) + \lambda_i g(x^i) \\ &= (1-\lambda_i)(g(\hat{x}) - \mu) + \mu + \lambda_i(\mu_i - \mu) \\ &= \lambda_i(\beta + \delta_i) - \beta + \mu \le \mu - \epsilon_i < \mu, \end{aligned}$$

so that $(1-\lambda_i)\hat{x} + \lambda_i x^i \in \{x \mid g(x) < \mu\}$ with $(1-\lambda_i)\hat{x} + \lambda_i x^i \to \bar{x}$ which establishes the result. □

Theorem 11. *Let h and C be as given in (19). If $x \in \mathrm{dom}\,(\partial h)$ is such that $x \notin \arg\min h$, or equivalently, $0 \notin \partial h(x)$, then $N_{\mathrm{lev}_h(h(x))}(x) = \mathrm{cl}^*(\mathrm{cone}(\partial h(x)))$. In particular, we obtain that the Slater condition implies that the Abadie constraint qualification holds.*

Proof. Let $x \in \mathrm{dom}\,(\partial h)$. Since h is lower semi-continuous, the set $\mathrm{lev}_h\,(h(x))$ is closed. Moreover, since $\{y \mid h(y) < h(x)\}$ is nonempty, Lemma 4 implies that $\mathrm{cl}\left\{y h(y) < h(x)\right\} = \mathrm{lev}_h\,(h(x))$. Now, for $y \in \mathrm{dom}\,(h)$, $h'(x; y-x) < 0$ if and only if there is some $1 \ge \lambda > 0$ such that $h(x + \lambda(y-x)) < h(x)$ since

$$h'(x; y-x) = \inf_{t>0} \frac{h(x + t(y-x)) - h(x)}{t}.$$

Therefore, again by Lemma 4,

$$\begin{aligned} T_{\mathrm{lev}_h(h(x))}(x) &= \mathrm{cl}\,\{\lambda(z-x)\mid 0<\lambda,\ h(z)\le h(x)\} \\ &= \mathrm{cl}\,\{\lambda(z-x)\mid 0<\lambda,\ z\in \mathrm{cl}\,\{y\mid h(y)<h(x)\}\} \\ &= \mathrm{cl}\,\{\lambda(y-x)\mid 0<\lambda,\ h(y)<h(x)\} \\ &= \mathrm{cl}\,\left\{\lambda(y-x)\,\middle|\,0<\lambda,\ y\in \mathrm{dom}\,(h),\ h'(x;y-x)<0\right\}. \end{aligned}$$

Consequently,

$$N_{\mathrm{lev}_h(h(x))}(x) = \left\{z\,\middle|\,\langle z, y-x\rangle \le 0\ \forall\, y\in \mathrm{dom}\,(h) \text{ with } h'(x;y-x)<0\right\}.$$

In addition, we have $h'(x;\cdot): X\to \overline{\mathbb{R}}$ since $\partial h(x)\neq\emptyset$, so that Lemma 4 and [33, Theorem 11] combine to imply that $\mathrm{cl}\,\left(\left\{d\,\middle|\,h'(x;d)<0\right\}\right) = \left\{d\,\middle|\,\psi^*_{\partial h(x)}(d)\le 0\right\}$. Hence,

$$\begin{aligned} N_{\mathrm{lev}_h(h(x))}(x) &= \left\{z\,\middle|\,\begin{array}{c}\langle z,\lambda(y-x)\rangle\le 0\ \forall\,\lambda>0, \text{ and } y\in\mathrm{dom}\,(h)\\ \text{with } h'(x;\lambda(y-x))<0\end{array}\right\} \\ &= \left\{z\,\middle|\,\langle z,d\rangle\le 0\ \forall\, d \text{ with } h'(x;d)<0\right\} \\ &= \left\{d\,\middle|\,h'(x;d)<0\right\}^\circ \\ &= (\mathrm{cl}\,(\left\{d\,\middle|\,h'(x;d)<0\right\}))^\circ \\ &= \left\{d\,\middle|\,\psi^*_{\partial h(x)}(d)\le 0\right\}^\circ = \{d\mid\langle z,d\rangle\le 0\ \forall z\in\partial h(x)\}^\circ \\ &= \{d\mid\langle z,d\rangle\le 0\ \forall z\in \mathrm{cone}\,(\partial h(x))\}^\circ = \mathrm{cl}^*\,(\mathrm{cone}\,(\partial h(x))), \end{aligned}$$

where the first equality follows since $h'(x;\lambda(y-x)) = \lambda h'(x;y-x)$ for all $\lambda>0$, the second equality follows since $h'(x;d)=+\infty$ if $d\neq\lambda(y-x)$ with $\lambda>0$ and $y\in\mathrm{dom}\,(h)$, and the fourth equality follows since $K^\circ = \mathrm{cl}\,(K)^\circ$ for all $K\subset X$ convex. □

Theorem 12. *Let the hypotheses of Proposition 3 hold and suppose that* $\Sigma\subset$ *dom* (∂h) *and that the strong Slater condition* $0\notin\mathrm{cl}\,\left(\partial h(C\cap h^{-1}(0))\right)$ *holds. Then the global error bound (20) holds for some* $\alpha>0$ *if and only if the pair of sets* $\{C,\mathrm{lev}_h\,(0)\}$ *is linearly regular.*

Remark 10. The strong Slater condition $0\notin\mathrm{cl}\,\left(\partial h(C\cap h^{-1}(0))\right)$ implies that the subdifferential $\partial h(x)$ is uniformly bounded away from the origin on $C\cap h^{-1}(0)$. This condition is equivalent to the condition used in [22, Corollary 1, Part (b)] when $C=X$, but is weaker than the condition used in [22, Corollary 2, Part (b)]. Theorem 12 refines the implication (b) $\Rightarrow$ (a) in [22, Corollary 2] by characterizing precisely when a global error bound occurs under the strong Slater condition. Note also that the requirement that $C\cap h^{-1}(0)\subset\mathrm{int}\,(\mathrm{dom}\,(h))$, or equivalently $\Sigma\subset\mathrm{int}\,(\mathrm{dom}\,(h))$, in [22, Corollary 2] is replaced by the weaker condition $\Sigma\subset$ *dom* (∂h) in Theorem 12.

Proof. By Theorem 10, we need only show that the strong Slater condition and the linear regularity hypothesis implies the global error bound (20). First note that the strong Slater condition implies that the set $\{x\mid h(x)<0\}$ is nonempty, and so,

$$N_{\mathrm{lev}_h(0)}(x) = \mathrm{cl}^*\,(\mathrm{cone}\,(\partial h(x)))\quad \forall\, x\in h^{-1}(0)\cap\mathrm{dom}\,(\partial h). \tag{27}$$

Combining this with Lemma 3(iii) and Proposition 3, we find that

$$N_{\mathrm{lev}_h(0)}(x) = \mathrm{cl}^*(\mathrm{cone}(\partial h_+(x))) \quad \forall\, x \in \Sigma. \tag{28}$$

The strong Slater condition also implies that there is an $\alpha_1 > 0$ such that

$$\inf_{x\in C\cap h^{-1}(0)\cap \mathrm{dom}\,(\partial h)} \mathrm{dist}\,(0 \mid \partial h(x)) \geq \alpha_1,$$

and so, by [8, Lemma 3.4, Part 2],

$$\alpha_1 \mathbb{B}^\circ \cap \mathrm{cone}(\partial h_+(x)) \subset \partial h_+(x) \quad \forall\, x \in C \cap h^{-1}(0) \cap \mathrm{dom}\,(\partial h). \tag{29}$$

Setting $\alpha_2 = \min\{1, \alpha_1\}$, we obtain from (29) and Proposition 3 that

$$\alpha_2 \mathbb{B}^\circ \cap \mathrm{cone}(\partial h_+(x)) \subset \partial h_+(x) \quad \forall\, x \in \Sigma, \tag{30}$$

which in turn implies that

$$\alpha_2 \mathbb{B}^\circ \cap \mathrm{cl}^*(\mathrm{cone}(\partial h_+(x))) \subset \partial h_+(x) \quad \forall\, x \in \Sigma. \tag{31}$$

By Theorem 2, the pair of sets $\{C, \mathrm{lev}_h(0)\}$ is linearly regular if and only if there is an $\alpha_3 > 0$ such that

$$\alpha_3 \mathbb{B}^\circ \cap N_\Sigma(x) \subset \mathbb{B}^\circ \cap N_C(x) + \mathbb{B}^\circ \cap N_{\mathrm{lev}_h(0)}(x) \quad \forall\, x \in \Sigma,$$

since $\Sigma = C \cap \mathrm{lev}_h(0)$. By (28), this is equivalent to the statement

$$\alpha_3 \mathbb{B}^\circ \cap N_\Sigma(x) \subset \mathbb{B}^\circ \cap N_C(x) + \mathbb{B}^\circ \cap \mathrm{cl}^*(\mathrm{cone}(\partial h_+(x))) \quad \forall\, x \in \Sigma. \tag{32}$$

Consequently, (31) implies that for $\alpha = \alpha_2\alpha_3$

$$\alpha \mathbb{B}^\circ \cap N_\Sigma(x) \subset \mathbb{B}^\circ \cap N_C(x) + \partial h_+(x) \quad \forall\, x \in \Sigma, \tag{33}$$

which, by Theorem 8, implies that the global error bound (20) holds. □

It is easy to see that if the global error bound (20) holds for (19), then the function $f(x) = h_+(x) + \psi_C(x)$ has Σ as a set of weak sharp minima. The following result gives a sufficient condition for the reverse implication to hold, which is supplementary to [8, Theorem 7.1].

Theorem 13. *Let $h\colon X \mapsto \overline{\mathbb{R}}$ and C be as given in (19). Suppose that Σ is a set of weak sharp minima for h_+ over C, and that $\partial(h_+(x) + \psi_C(x)) = \partial h_+(x) + N_C(x)$ for all $x \in \Sigma$. If the set $\cup_{x\in\Sigma}\partial h_+(x)$ is bounded, then the global error bound (20) holds.*

Proof. Since Σ is a set of weak sharp minima for h_+ over C, there is an $\alpha > 0$ such that

$$\alpha \mathbb{B}^\circ \cap N_\Sigma(x) \subset \partial h_+(x) + N_C(x) \quad \forall\, x \in \Sigma. \tag{34}$$

By assumption, there is a $\beta > 0$ such that

$$\|x^*\| \leq \beta \quad \forall\, x^* \in \cup_{x\in\Sigma}\partial h_+(x). \tag{35}$$

The inclusion (34) along with (35) implies that

$$\alpha\mathbb{B}^\circ \cap N_\Sigma(x) \subset \partial h_+(x) + (\alpha+\beta)\mathbb{B}^\circ \cap N_C(x) \quad \forall\, x \in \Sigma, \tag{36}$$

which in turn implies that, for any $x \in \Sigma$,

$$\alpha\mathbb{B}^\circ \cap N_\Sigma(x) \subset \begin{cases} \partial h_+(x) + \mathbb{B}^\circ \cap N_C(x) & , \text{ if } \alpha+\beta \le 1, \\ (\alpha+\beta)\partial h_+(x) + (\alpha+\beta)\mathbb{B}^\circ \cap N_C(x) & , \text{ if } \alpha+\beta > 1. \end{cases}$$

This completes the proof. □

We now consider extensions to systems of the form (22). Let us first consider the case where the index set T is finite: $T = \{1, 2, \ldots, N\}$. As was the case for the function ρ defined in expression (7) of Section 3, the equivalence of norms in finite dimensions implies that the error bound (38) is equivalent to any error bound of the form

$$\tilde{\alpha}\mathrm{dist}\,(x \mid \Sigma) \le \left\| \left(\mathrm{dist}\,(x \mid C), h_+^1(x), \ldots, h_+^N(x)\right)^T \right\|, \tag{37}$$

where $h_+^i(x) = \max\{0, h^i(x)\}$ $i = 1, \ldots, N$. If the norm $\|\cdot\|$ is monotone, then the function

$$\bar{\rho}(x) = \left\| \left(\mathrm{dist}\,(x \mid C), h_+^1(x), \ldots, h_+^N(x)\right)^T \right\|$$

is convex. For example, one can use the ∞-norm on $\mathbb{R}^{N+1}$ to obtain the types of error bounds discussed in [22]. Note that since we deal with only finitely many inequalities, results obtained for one choice of norm are easily translated into a result for another choice of norm using the equivalence of norms. When one extends the ideas presented here to infinite index sets, then one must first embed the problem into an appropriate function space and then be careful to respect the geometry of that space. In our discussion of (37) we choose the 1-norm and consider weak sharp minima for the function

$$\alpha\mathrm{dist}\,(x \mid \Sigma) \le \mathrm{dist}\,(x \mid C) + \sum_{i=1}^N h_+^i(x). \tag{38}$$

Weak sharp minima characterization readily flow from the earlier results in this section by simply computing the subdifferential of the function $\rho_1(x) = \sum_{i=1}^N h_+^i(x)$. Rather than pursuing all of the consequences of Theorems 8 and 9 in this context, we instead focus on the link between the existence of an error bound for the system (22) and the linear regularity of the level sets of the functions h^i, $i = 1, \ldots, N$. We begin with the following definition.

Definition 5. *Let $h^i : X \mapsto \overline{\mathbb{R}}$, $i = 1, \ldots, N$ be as given in (22). Define the operator $D\colon X \mapsto \prod_{i=1}^N X^*$ by $D = \partial h_+^1 \times \partial h_+^2 \times \cdots \times \partial h_+^N$, and set*

$$\begin{aligned} D(\Sigma) &= \left\{ (z^1, z^2, \ldots, z^N) \in D(x) \mid x \in \Sigma \right\} \\ &= \left\{ (z^1, z^2, \ldots, z^N) \;\middle|\; x \in \Sigma,\ z^i \in \partial h_+^i(x)\, i = 1, 2, \ldots, N \right\}. \end{aligned}$$

We say that the system (22) is asymptotically additively regular if for every $(w^1, w^2, \ldots, w^N)$ in $D(\Sigma)^\infty$ satisfying $\sum_{i=1}^N w^i = 0$, we have $w^i = 0$, $i = 1, 2, \ldots, N$.

Theorem 14. *Let X be a finite dimensional space and consider the system (22). Assume that $\Sigma \subset \cap_{i=1}^{N} \text{int}\,(\text{dom}\, h^i)$. Let $h(x) = \sum_{i=1}^{N} h_+^i(x)$ and $C_i = \{x \mid h^i(x) \leq 0\}$ for $i = 1, 2, \ldots, N$. Consider the following statements:*

1. *The set Σ is a set of boundedly weak sharp minima for h over C.*
2. *The set Σ is a set of boundedly weak sharp minima for $h + \text{dist}\,(\cdot \mid C)$.*
3. *The collection $\{C, C_1, C_2, \ldots, C_N\}$ is boundedly linearly regular.*

Statements 1 and 2 are equivalent, and they imply statement 3. Conversely, if for each $i = 1, 2, \ldots, N$, the set C_i is a set of boundedly weak sharp minima for h_+^i, then statement 3 implies statements 1 and 2. In addition, if Σ is a set of weak sharp minima for $h + \text{dist}\,(\cdot \mid C)$ and the system (22) is asymptotically additively regular, then the collection $\{C, C_1, C_2, \ldots, C_N\}$ is linearly regular.

Proof. The equivalence of statements 1 and 2 follows from [8, Theorem 7.1] since h is Lipschitz continuous on any bounded subsets of X. We now show that statement 2 implies that the collection of sets $\{C, C_1, \ldots, C_N\}$ is boundedly linearly regular. By [8, Corollary 5.3] and [8, Theorem 6.3], we have that, for any $r > 0$, there is an $\alpha(r) > 0$ such that

$$\alpha(r)\mathbb{B}^\circ \cap N_\Sigma\,(x) \subset \partial h(x) + \mathbb{B}^\circ \cap N_C\,(x) \quad \forall x \in \Sigma \cap r\mathbb{B} \tag{39}$$

since $\partial(h(x) + \text{dist}\,(x \mid C)) = \partial h(x) + \mathbb{B}^\circ \cap N_C\,(x)$. For any $x \in \Sigma$, cone $(\partial h_+^i(x)) \subset N_{C_i}\,(x)$ for $i = 1, 2, \ldots, N$. By $\Sigma \subset \text{int}\,(\text{dom}\, h^i)$, the set $\cup_{x \in \Sigma \cap r\mathbb{B}} \partial h_+^i(x)$ is a bounded set. Therefore, there is a $\beta(r) > 0$ such that

$$\partial h_+^i(x) \subset \beta(r)\mathbb{B}^\circ \cap N_{C_i}\,(x) \quad \forall x \in \Sigma \cap r\mathbb{B} \tag{40}$$

for $i = 1, 2, \ldots, N$. Inclusions (39) and (40) along with the fact that $\partial h(x) = \sum_{i=1}^{N} \partial h_+^i(x)$ yield

$$\alpha(r)\mathbb{B}^\circ \cap N_\Sigma\,(x) \subset (\sum_{i=1}^{N} \beta(r)\mathbb{B}^\circ \cap N_{C_i}\,(x)) + \mathbb{B}^\circ \cap N_C\,(x) \quad \forall x \in \Sigma \cap r\mathbb{B}.$$

This shows that the collection $\{C, C_1, C_2, \ldots, C_N,\}$ is boundedly linearly regular.

For the converse, suppose that statement 3 along with the hypotheses hold. Then, for any $r > 0$, there are an $\alpha(r) > 0$ and $\beta^i(r)$ for $i = 1, 2, \ldots, N$ such that

$$\alpha(r)\mathbb{B}^\circ \cap N_\Sigma\,(x) \subset (\sum_{i=1}^{N} \mathbb{B}^\circ \cap N_{C_i}\,(x)) + \mathbb{B}^\circ \cap N_C\,(x) \quad \forall x \in \Sigma \cap r\mathbb{B}, \tag{41}$$

and

$$\mathbb{B}^\circ \cap N_{C_i}\,(x) \subset \beta^i(r)\mathbb{B}^\circ \cap \partial h_+^i(x) \quad \forall x \in C_i \cap r\mathbb{B}. \tag{42}$$

Let $\beta(r) = \max\{\beta^i(r) \mid i = 1, 2, \ldots, N\}$. Then the inclusions (41) and (42) yield

$$\alpha(r)\mathbb{B}^\circ \cap N_\Sigma\,(x) \subset (\sum_{i=1}^{N} \beta(r)\mathbb{B}^\circ \cap \partial h_+^i(x)) + \mathbb{B}^\circ \cap N_C\,(x) \quad \forall x \in \Sigma \cap r\mathbb{B},$$

which shows that statement 2 holds.

Let us now suppose that (20) holds, and the system (22) is asymptotically additively regular. Then there is an $\alpha > 0$ such that

$$\alpha \mathbb{B}^\circ \cap N_\Sigma(x) \subset \partial h(x) + \mathbb{B}^\circ \cap N_C(x) = \sum_{i=1}^{N} \partial h_+^i(x) + \mathbb{B}^\circ \cap N_C(x) \quad \forall x \in \Sigma. \tag{43}$$

It follows that

$$\alpha \mathbb{B}^\circ \cap N_\Sigma(x) \subset (1+\alpha)\mathbb{B}^\circ \cap (\sum_{i=1}^{N} \partial h_+^i(x)) + \mathbb{B}^\circ \cap N_C(x) \quad \forall x \in \Sigma. \tag{44}$$

We claim that there is a $\beta > 0$ such that

$$(1+\alpha)\mathbb{B}^\circ \cap (\sum_{i=1}^{N} \partial h_+^i(x)) \subset \sum_{i=1}^{N} (\beta \mathbb{B}^\circ \cap \partial h_+^i(x)) \quad \forall x \in \Sigma. \tag{45}$$

If the inclusion (45) were not true, then, since $\Sigma \subset \cap_{i=1}^N$ int dom (h^i), there would be a sequence $\{x^j\} \subset \Sigma$ and $z^i(j) \in \partial h_+^i(x^j)$ with $\max\{\|z^i(j)\| \mid i = 1, 2, \ldots, N\} \to \infty$ as $j \to \infty$, and $\left\|\sum_{i=1}^N z^i(j)\right\| \le 1+\alpha$ for all j. Without loss of generality, suppose that $\max_{i=1}^N\{\|z^i(j)\|\} = \|z^1(j)\|$ for $j = 1, 2, \ldots,$ and $z^i(j)/\|z^1(j)\| \to w^i$, $i = 1, 2, \ldots, N$ with $\|w^1\| = 1$. By construction $(w^1, w^2, \ldots, w^N) \in D(\Sigma)^\infty$ and $\sum_{i=1}^N w^i = 0$, but $w^1 \neq 0$. That is, the system (22) is not asymptotically additively regular. This contradiction establishes the inclusion (45). By (43), (44) and (45) and the fact that cone $(\partial h_+^i(x)) \subset N_{C_i}(x)$ for all $x \in \Sigma$ and $i = 1, 2, \ldots, N$, we have $\alpha \mathbb{B}^\circ \cap N_\Sigma(x) \subset \sum_{i=1}^N (\beta \mathbb{B}^\circ \cap N_{C_i}(x)) + \mathbb{B}^\circ \cap N_C(x)$, $\quad \forall x \in \Sigma$. This proves that the collection $\{C, C_1, C_2, \ldots, C_N\}$ is linearly regular. □

Remark 11. The set C_i is a set of boundedly weak sharp minima for h_+^i whenever the Slater condition holds for the inequality system $h^i(x) \le 0$.

Remark 12. A simple sufficient condition for the asymptotic additive regularity condition is the strong additive regularity of the collection

$$\{\partial h_+^1(\Sigma)^\infty, \partial h_+^2(\Sigma)^\infty, \ldots, \partial h_+^N(\Sigma)^\infty\}.$$

Also since $\Sigma \subset \cap_{i=1}^N$int (dom h^i), the strong additive regularity of the system (22) holds trivially if either the solution set Σ is bounded or $N = 1$.

Remark 13. Note that if $x \in \Sigma$ satisfies $h^i(x) < 0$, then $\partial h_+^i(x) = \{0\}$. Hence the asymptotic constraint qualification introduced by Auslender and Crouzeix [2] for the system (22) implies that the system (22) is asymptotically additively regular. Unlike the asymptotic constraint qualification [2], the asymptotic additive regularity condition and the weak Slater condition [17, Definition VII. 2.2.3] combined do not imply the global error bound (20) for the system (22); for instance, the inequality system $\sqrt{x_1^2 + x_2^2} - x_1 - 1 \le 0$ satisfies the Slater condition, and the asymptotic additive regularity condition holds trivially, but (20) fails.

Remark 14. When $C = X$, the collection of $\{C_1, C_2, \dots, C_N\}$ is linear regular if the weak Slater condition and the asymptotic constraint qualification both hold [5, Theorem 8]. On the other hand, it is well-known in the error bounds literature that the weak Slater condition and the asymptotic constraint qualification together imply that the set $\sum$ is a set of weak sharp minima for $h + \operatorname{dist}(\cdot \mid C)$. Therefore, the comments in Remark 3 above show that the last part of Theorem 14 is a refinement of [5, Theorem 8].

The pattern of proof described above for the system (22) can be extended to the case of infinite index sets by making the appropriate definitions and restriction. Results of this type have been established in [23] and [39]. We do not pursue this line here, rather we consider a different kind of extension to infinite index sets which appears in many applications. In this setting the index set T has topological structure and the functions $h_t(x) = h(t, x)$ are continuous functions of the index t. Again, we only consider a sample result.

Following Clarke [11], we make the following additional assumptions.

(A1) T is metrizable and sequentially compact.
(A2) The functions $h_t\colon X \mapsto \mathbb{R}$ are finite-valued, continuous, and convex for each $t \in T$.
(A2) The mapping $t \to h_t(x)$ is continuous for each $x \in X$.
(A3) For each $x \in X$, there is a neighborhood U of x such that the functions $\{ h_t \mid t \in T \}$ are uniformly Lipschitz continuous on U.
(A4) For each $x \in X$, the set $\{ h_t(x) \mid t \in T \}$ is bounded.

With these assumptions Clarke [11, Theorem 2.8.2] shows that for each $x \in X$ the subdifferential of the convex function

$$h(x) = \sup_{t \in T} h_t(x)$$

is given by the formula

$$\partial h(x) = \left\{ \int_T \partial h_t(x)\mu(dt) \mid \mu \in P\,[M(x)] \right\},$$

where $M(x) = \{ t \in T \mid h_t(x) = h(x) \}$ and $P\,[M(x)]$ is the set of Radon probability measures supported on $M(x)$. With this formula we immediately obtain the following characterization of the existence of a global error bound of the form (21) for the system (22). Again no proof is required since the result follows immediately from Proposition 3 and Theorem 9.

Theorem 15. *Let the functions h_t, $t \in T$ satisfy the hypotheses (A1)-(A4) and suppose that the Slater constraint qualification $\{ x \in C \mid h(x) < 0 \} \neq \emptyset$ is satisfied. Then the system (22) satisfies the global error bound (21) if and only if there is an $\alpha > 0$ such that for all $x \in C \cap h^{-1}(0)$,*

$$\alpha \mathbb{B}^\circ \cap N_\Sigma(x) \subset \mathbb{B}^\circ \cap N_C(h(x)) + \mathrm{cl}^* \left\{ \int_T \partial h_t(x)\mu(dt) \mid \mu \in P_-\,[M(x)] \right\},$$

where $P_-\,[M(x)]$ is the set of non-negative Radon measures μ supported on $M(x)$ for which $\mu(M(x)) \le 1$ and the integral of a multifunction is defined to be the integral of all measurable selections from $\partial h_t(x)$ [11, 34].

Error bounds involving integral functionals of the form

$$h(x) = \int_T h_+(t, x)\, dt$$

can be derived in a similar fashion using Rockafellar's theory of normal integrands [34]. However, such a development would take us too far afield of the central theme of this paper.

A. Barrier cone properties

During the refereeing process, a error was discovered in [8, Lemma B.3, Formula (B.8)]. This error does not change the correctness of any other result in [8]. The corrected version of [8, Lemma (B.3)] is given below.

Lemma 5. *Let C be a nonempty closed convex subset of normed linear space X and set $K = \bigcup_{x\in C} N_C\,(x)$. Then*

$$K = \mathrm{dom}\ \left(\partial\psi_C^*\right) \subset \mathrm{dom}\ \left(\psi_C^*\right) = \mathrm{bar}\ (C) \tag{A.1}$$

and

$$\mathrm{cl}^*\,(K) \subset \mathrm{cl}^*\,(\mathrm{bar}\ (C)) = (C^\infty)^\circ. \tag{A.2}$$

In particular,

$$(K)^\circ \supset C^\infty. \tag{A.3}$$

If X is assumed to be Banach, then

$$\mathrm{w-cl}\,(K) = \mathrm{cl}\,(K) = \mathrm{cl}\,(\mathrm{bar}\ (C)) \subset \mathrm{cl}^*\,(K) = \mathrm{cl}^*\,(\mathrm{bar}\ (C)) = (C^\infty)^\circ. \tag{A.4}$$

If X is assumed to be reflexive, we obtain equality throughout (A.3) and (A.4). If it is further assumed that X is finite dimensional, then

$$\mathrm{ri}\ (\mathrm{bar}\ (C)) \subset K. \tag{A.5}$$

Proof. Since $\partial\psi_C = N_C$, we obtain from [7, Theorem 2] (or [38, Theorem 3.1.2]) that

$$z \in N_C\,(x) \iff x \in \partial\psi_C^*(z).$$

The relations (A.1), (A.2), and (A.3) immediately follow.

Assume X is Banach, then, by [7, Lemma 1, p608] (or [38, Theorem 3.1.2]), we have

$$\mathrm{bar}\ (C) \subset \mathrm{cl}\,(K) \tag{A.6}$$

since the range of the subdifferential mapping $\partial\psi_C$ satisfies $\mathrm{Ran}\,(\partial\psi_C) = K$. Since $\mathrm{bar}\ (C)$ is convex (it is the domain of a convex function), $\mathrm{cl}(\mathrm{bar}\ (C)) = \mathrm{w-cl}(\mathrm{bar}\ (C))$ [13, Theorem 1]. Hence the first two equivalences in (A.4) follow since, by the inclusions (A.1) and (A.6), we have

$$\mathrm{cl}(\mathrm{bar}\ (C)) = \mathrm{cl}(K) \subset \mathrm{w-cl}(K) \subset \mathrm{w-cl}(\mathrm{bar}\ (C)) = \mathrm{cl}(\mathrm{bar}\ (C)).$$

The inclusion in (A.4) is obvious. The third equivalence in (A.4) follows from (A.1) and

$$\text{bar }(C) \subset \text{cl}(\text{bar }(C)) = \text{cl}(K) \subset \text{cl}^* (K).$$

The final equivalence is given in [31, Page 50]. This equivalence can also be viewed as a special case of formula (B.2) in [8, Lemma B.1].

When X is reflexive, $\text{cl}\,(\text{bar }(C)) = \text{cl}^*\,(\text{bar }(C))$, and so all sets in (A.4) are equivalent. This in turn implies the equivalence of the sets in (A.3). The final inclusion (A.5) is an immediate consequence of [32, Theorem 23.4]. □

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