A Non–Interior Predictor–Corrector Path Following Algorithm for the Monotone Linear Complementarity Problem*

Jim Burke and Song Xu
Department of Mathematics
University of Washington
Seattle WA 98195.

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Abstract

We present a predictor–corrector non–interior path following algorithm for the monotone linear complementarity problem based on Chen–Harker–Kanzow–Smale smoothing techniques. Although the method is modeled on the interior point predictor–corrector strategies, it is the first instance of a non–interior point predictor–corrector algorithm. The algorithm is shown to be both globally linearly convergent and locally quadratically convergent under standard hypotheses. The approach to global linear convergence follows the authors' previous work on this problem for the case of $(P_0 + R_0)$ LCPs. However, in this paper we use monotonicity to refine our notion of neighborhood of the central path. The refined neighborhood allows us to establish the uniform boundedness of certain slices of the neighborhood of the central path under the standard hypothesis that a strictly positive feasible point exists.

1 Introduction

We consider a path–following algorithm for the *monotone linear complementarity problem*:

LCP(q, M): Find $(x, y) \in I\!R^n \times I\!R^n$ satisfying

$$Mx - y + q = 0, \tag{1.1}$$
$$x \geq 0, y \geq 0, x^T y = 0, \tag{1.2}$$

where $M \in I\!R^{n\times n}$ is positive semi-definite and $q \in I\!R^n$.

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Here the path to be followed is the *central path*

$$\mathcal{C} = \{(x, y) \,:\, 0 < \mu,\ 0 < x,\ 0 < y,\ Mx - y + q = 0, \text{ and } Xy = \mu^2 e\} \tag{1.3}$$

where, following standard usage in the interior-point literature [15], we denote by $e \in I\!R^n$ the vector each of whose components is 1 and by X the diagonal matrix whose diagonal entries are given by the vector $x \in I\!R^n$. The algorithm is based on Chen-Harker-Kanzow-Smale smoothing techniques [4, 13, 21] and as such relies on the function

$$\phi(a, b, \mu) = a + b - \sqrt{(a - b)^2 + 4\mu^2}\ . \tag{1.4}$$

This function is a member of the Chen–Mangasarian class of smoothing functions for the problem LCP(q, M) [6]. It is easily verified that for $\mu > 0$

$$\phi(a, b, \mu) = 0 \text{ if and only if } 0 < a,\ 0 < b, \text{ and } ab = \mu^2. \tag{1.5}$$

As with all path following algorithms for LCP(q, M), the central idea is to use Newton's method to track the central path for decreasing values of the smoothing parameter μ. The various path following methods are distinguished by the systems of equations used to identify the central path, or more generally, a *smoothing path*, and the neighborhood used to control the deviation from the central path and to update the smoothing parameter. Regarding the system of equations, we follow the pattern first suggested in the non–interior point context by Hotta and Yoshise [10] (also see Qi and Sun [18]) and include the smoothing parameter μ in the set of parameters to which the Newton iteration is applied. However, our choice of neighborhood for the central path differs markedly from that studied in [10, 18]. In addition, we develop a non–interior predictor–corrector strategy for following the central path. Our approach is modeled on the predictor–corrector methodology commonly employed in the interior point literature [16, 24].

The algorithm given in this paper is the first non–interior predictor–corrector strategy proposed for following the central path. The central idea is to apply Newton's method to equations of the form $F(x, y, \mu) = v$ for various choices of the right hand side v where the function $F : I\!R^n \times I\!R^n \times I\!R_+ \to I\!R^n \times I\!R^n \times I\!R_+$ is given by

$$F(x, y, \mu) := \begin{bmatrix} Mx - y + q \\ \Phi(x, y, \mu) \\ \mu \end{bmatrix}\ , \tag{1.6}$$

with

$$\Phi(x, y, \mu) = \begin{bmatrix} \phi(x_1, y_1, \mu) \\ \cdots \\ \phi(x_n, y_n, \mu) \end{bmatrix}. \tag{1.7}$$

Note that

$$F(x, y, \mu) = 0 \tag{1.8}$$

if and only if (x, y) solves LCP(q, M), and

$$F(x, y, \mu) = \begin{bmatrix} 0 \\ 0 \\ \bar{\mu} \end{bmatrix} \quad \text{with } \ \bar{\mu} \neq 0 \tag{1.9}$$

if and only if (x, y) is on the central path $\mathcal{C}$ with corresponding smoothing parameter $\bar{\mu}$.

The general outline of our predictor–corrector algorithm can now be described. First, a predictor step is computed. This is the Newton step based on the equation (1.8) at the current iterate. The predictor step in the x and y variables is either accepted or rejected depending on whether it is in a pre–specified neighborhood of the central path with the current value of the smoothing parameter. If it is in this neighborhood, then the predictor step is accepted and a backtracking routine is applied to the smoothing parameter μ to reduce its value as much as is possible subject to remaining in the neighborhood of the central path. Next, a corrector step is computed. This is the Newton step based on the equation (1.9) at the iterate obtained from the predictor step with $\bar{\mu}$ taken to be a fixed fraction of the value for μ obtained from the predictor step. Again, a step length is determined to return the update to the neighborhood of the central path. The technique for choosing the corrector step guarantees the global linear convergence of the method, while the technique for choosing the predictor step provides for the local quadratic convergence of the iterates. The line search routines for both the predictor and corrector steps are based on finitely terminating backtracking procedures and as such are easily implemented.

A number of non–interior path following algorithms have recently been proposed that are globally convergent or globally linearly convergent and possess rapid local convergence properties [2, 3, 4, 5, 7, 8, 10, 11, 13, 18, 19, 23, 25, 26]. The papers [7, 8, 11, 19] are not path following algorithms and do not establish global rates of convergence. In [7], Chen, Qi, and Sun consider smoothing methods for box constrained variational inequalities and established the global convergence and the local super–linear convergence of their *Smoothing Newton Method.* Chen and Ye [8] build on the work in [7] and develop a hybrid smoothing Newton method that is globally convergent, locally super–linear, and possesses a finite termination property for linear variational inequality problems. Jiang [11] develops a generalized Newton and Gauss-Newton methods for the complementarity problem. He establishes both the global convergence and local super–linear convergence of each method. In [19], Qi, Sun, and Zhou apply techniques from non–smooth equations to obtain the global convergence and local super–linear convergence of a smoothing method based on Robinson's normal equations [20].

The first non–interior path following method for LCP was developed by Chen and Harker [4]. This method and other closely related methods were further studied by Kanzow [13]. However, no rate of convergence is established in either [4] or [13].

The concept of a neighborhood for the central path, common in the interior point literature, is brought to bear on non–interior path following algorithms for the first time in [1]. By doing so, Burke and Xu are able to obtain the first global linear convergence result for non–interior path following methods. Their results apply to LCPs whose affine equation is determined by a matrix that is both a P_0 and an R_0 matrix. Xu [25] develops an infeasible non-interior path-following method for nonlinear complementarity problems based on uniform P-functions. Using a more general notion of neighborhood, he establishes the first global linear convergence result for non–linear complementarity problems using non–interior path following ideas. Further results concerning boundedness properties for neighborhoods of the central path, stopping criterion, and complexity of the non-interior path-following method for monotone LCPs are given by Xu in [26].

A different notion of neighborhood for the central path for monotone NCPs is introduced in [10] by Hotta and Yoshise. They also study some structural properties of non–interior smoothing methods and propose an algorithm for which they are able to establish a global convergence result. In [18], Qi and Sun develop a non–interior path following algorithm using the neighborhood ideas developed by Hotta and Yoshise [10]. Conditions are given under which the algorithm is globally linearly convergent, or globally convergent and locally super–linearly convergent.

The papers [2, 3, 5] modify the neighborhood concepts introduced in [1, 10] and establish the global linear convergence of their non–interior point path following algorithms. In addition, they introduce the idea of an *Approximate Newton Step* to obtain local quadratic or super–linear convergence. In [5], Chen and Xiu compute both a centering step and an approximate Newton step based on a single matrix factorization. If the approximate Newton step performs better than the centering step, then the new iterate is based on the approximate Newton step. In [2], only the approximate Newton step is used but two backtracking line searches are required to obtain both global linear and local super–linear convergence. In [3], Chen and Chen use a new technique for dynamically updating the neighborhood of the central path in order to establish global convergence and local super–linear convergence.

In [23], Tseng combines proof techniques developed in [22] for infeasible interior point methods with the neighborhood ideas for non–interior point methods to obtain a global linear convergence for infeasible non–interior path following methods. The search direction is a combination of a centering step with a step designed to accelerate the method locally. In addition, an active set strategy is introduced that allows one to establish the local super–linear convergence of the method under very mild conditions.

The plan of the paper is as follows. In Section 2 we introduce our refined neighborhood for the central path and establish the uniform boundedness of certain slices of this neighborhood when it is assumed that LCP(q, M) is monotone and has a feasible interior point. In Section 3, we state our predictor–corrector algorithm and show that it is well–defined. Finally, Section 4 contains the convergence analysis.

A few words about our notation are in order. All vectors are column vectors with the superscript T denoting transpose. The notation $I\!R^n$ is used for real n–dimensional space and $I\!R^{n\times n}$ is used to denote the set of all $n \times n$ real matrices. We denote the positive orthant in $I\!R^n$ by $I\!R^n_+$. Given $x, y \in I\!R^n$, we write $x \le y$ to indicate that $y - x \in I\!R^n_+$. Given $x \in I\!R^n$, we denote by $\|x\|_1$, $\|x\|$, and $\|x\|_\infty$, the 1–norm, 2–norm, and ∞–norm of x, respectively. A matrix $M \in I\!R^{n\times n}$ is said to be a P_0 matrix if all of its principal minors are non–negative.

2 A Neighborhood of the Central Path

We take the set

$$\mathcal{N}(\beta) := \left\{ (x, y) \;\middle|\; \begin{array}{l} Mx - y + q = 0,\ \Phi(x, y, \mu) \le 0, \\ \|\Phi(x, y, \mu)\| \le \beta\mu \text{ for some } \mu > 0 \end{array} \right\} , \tag{2.10}$$

as our neighborhood of the central path, where $\beta > 0$ is given. This neighborhood can be viewed as the union of the *slices*

$$\mathcal{N}(\beta,\mu) := \{ (x,y) \,:\, Mx - y + q = 0,\ \Phi(x,y,\mu) \le 0,\ \|\Phi(x,y,\mu)\| \le \beta\mu\} \tag{2.11}$$

for $\mu > 0$. This neighborhood (2.11) refines the neighborhood concept introduced in [1] by requiring that all points in the neighborhood satisfy the inequality $\Phi(x,y,\mu) \le 0$. Under the assumption of monotonicity plus the existence of a strictly feasible interior point, this refinement guarantees the boundedness of the sets $\cup_{0<\mu\le\mu_0}\mathcal{N}(\beta,\mu)$ for any choice of $\beta > 0$ and $\mu_0 > 0$ (see Lemma 2.1). In addition, it is shown in Theorem 3.1 that the componentwise concavity of the function Φ (Lemma 2.2) implies that if the algorithm is initiated at a point in this neighborhood, then subsequent iterates automatically satisfy the inequality $\Phi(x,y,\mu) \le 0$.

The algorithm is constructed so that each iterate is contained in a slice and that the associated values of μ can be driven to zero at a linear rate. It then remains to show that any cluster point of such a sequence solves LCP(q,M). The existence of such cluster points is established by showing that the slices are uniformly bounded for $0 < \mu \le \mu_0$ for any choice of $\beta > 0$ and $\mu_0 > 0$.

Assumption (A):

The problem LCP(q,M) is monotone and has a feasible interior point $(x,y) \in \mathbb{R}^{n\times n}$, i.e.,

$$x > 0, y > 0 \text{ and } Mx - y + q = 0.$$

It is well–known [15] that Assumption (A) is sufficient to establish the existence of the central path and the existence of a solution to LCP(q,M). We now show that it also suffices to establish the uniform boundedness of the slices.

Lemma 2.1 *Assume that condition (A) holds. Then for any $\beta > 0$ and $\mu_0 > 0$, the set*

$$\cup_{0<\mu\le\mu_0}\mathcal{N}(\beta,\mu)$$

is bounded. Indeed, for any $(x,y) \in \cup_{0<\mu\le\mu_0}\mathcal{N}(\beta,\mu)$, we have for $i = 1,2,\ldots,n$

$$\begin{aligned}
-(\beta\mu_0)/2 \le x_i \quad &\le \quad \frac{\bar{x}^T\bar{y} + \frac{\beta\mu_0}{2}(\|\bar{x}\|_1 + \|\bar{y}\|_1) + n\max\{\mu_0^2, \frac{\beta^2\mu_0^2}{4}\}}{\bar{y}_j} \\
-(\beta\mu_0)/2 \le y_i \quad &\le \quad \frac{\bar{x}^T\bar{y} + \frac{\beta\mu_0}{2}(\|\bar{x}\|_1 + \|\bar{y}\|_1) + n\max\{\mu_0^2, \frac{\beta^2\mu_0^2}{4}\}}{\bar{x}_i},
\end{aligned}$$

where $(\bar{x},\bar{y})$ is any feasible interior point, that is, a point satisfying

$$M\bar{x} - \bar{y} + q = 0, \quad \bar{x} > 0,\ \bar{y} > 0.$$

Proof Let $\beta > 0$, $0 < \mu \leq \mu_0$, and $(x,y) \in \mathcal{N}(\beta,\mu)$ be given, and let $(\bar{x},\bar{y})$ be a feasible interior point for LCP(q,M). First observe that if $-\delta \leq \phi(a,b,\mu)$, then $-\delta/2 < \min\{a,b\}$. To see this, note that the condition $-\delta \leq \phi(a,b,\mu)$ implies that

$$0 < \sqrt{[(a+\delta/2)-(b+\delta/2)]^2 + 4\mu^2} \leq (a+\delta/2) + (b+\delta/2). \tag{2.12}$$

Squaring both sides and cleaning up yields $0 < \mu^2 \leq (a+\delta/2)(b+\delta/2)$. Thus, since at least one of $(a+\delta/2)$ and $(b+\delta/2)$ must be positive by (2.12), both must be positive yielding $-\delta/2 < \min\{a,b\}$. This observation implies that

$$x_i > -(\beta\mu)/2 \geq -(\beta\mu_0)/2, \text{ and } y_i > -(\beta\mu)/2 \geq -(\beta\mu_0)/2, \text{ for } i = 1,2,\ldots,n\ . \tag{2.13}$$

Next, note that if $0 \leq a$ and $0 \leq b$, then the inequality $\phi(a,b,\mu) \leq 0$ implies that $0 \leq a+b \leq \sqrt{(a-b)^2+4\mu^2}$. Again, by squaring and cleaning up, we see that this gives $ab \leq \mu^2$. This observation implies that

$$x_i y_i \leq \mu_0^2 \text{ for each } i \in \{1,\ldots,n\} \text{ with } 0 < x_i,\ 0 < y_i\ . \tag{2.14}$$

We conclude the proof by noting that monotonicity yields $0 \leq (\bar{x}-x)^T(\bar{y}-y)$, or equivalently $\bar{x}^T y + \bar{y}^T x \leq \bar{x}^T\bar{y} + x^T y$. This inequality plus those in (2.13) and (2.14) yield the bound

$$\begin{aligned}
\sum_{y_i>0} \bar{x}_i y_i + \sum_{x_i>0} \bar{y}_i x_i &\leq \bar{x}^T\bar{y} + x^T y - \left[\sum_{y_i<0} \bar{x}_i y_i + \sum_{x_i<0} \bar{y}_i x_i\right] \\
&\leq \bar{x}^T\bar{y} + x^T y + \frac{\beta\mu_0}{2}(\|\bar{x}\|_1 + \|\bar{y}\|_1) \\
&\leq \bar{x}^T\bar{y} + \sum_{\substack{x_i>0\\y_i>0}} x_i y_i + \sum_{\substack{x_i<0\\y_i<0}} x_i y_i + \frac{\beta\mu_0}{2}(\|\bar{x}\|_1 + \|\bar{y}\|_1) \\
&\leq \bar{x}^T\bar{y} + n\max\{\mu_0^2, \frac{\beta^2\mu_0^2}{4}\} + \frac{\beta\mu_0}{2}(\|\bar{x}\|_1 + \|\bar{y}\|_1)\ .
\end{aligned}$$

It follows that if $y_i > 0$, then

$$y_i \leq \frac{\bar{x}^T\bar{y} + \frac{\beta\mu_0}{2}(\|\bar{x}\|_1 + \|\bar{y}\|_1) + n\max\{\mu_0^2, \frac{\beta^2\mu_0^2}{4}\}}{\bar{x}_i},$$

and, if $x_i > 0$, then

$$x_i \leq \frac{\bar{x}^T\bar{y} + \frac{\beta\mu_0}{2}(\|\bar{x}\|_1 + \|\bar{y}\|_1) + n\max\{\mu_0^2, \frac{\beta^2\mu_0^2}{4}\}}{\bar{y}_i}.$$

□

We conclude this section by cataloging a few technical properties of the function $\phi(a,b,\mu)$ for later use.

Lemma 2.2 *The function ϕ defined in (1.4) has the following properties:*

(i) *[13] The function $\phi(a,b,\mu)$ is continuously differentiable on $I\!R^2 \times I\!R_+$.*

(ii) *The function $\phi(a,b,\mu)$ is concave on $I\!R^2 \times I\!R_+$,*

(iii) *[18, Lemma 2] For any $(a,b,\mu) \in I\!R^2 \times R_+$, we have*

$$\left\|\nabla^2\phi(a,b,\mu)\right\| \le \frac{4}{\sqrt{(a-b)^2+4\mu^2}} \le \frac{2}{\mu}.$$

Proof (ii): It is easy to check that

$$\nabla\phi(a,b,\mu) = \begin{pmatrix} 1 - \frac{a-b}{\sqrt{(a-b)^2+4\mu^2}} \\ 1 + \frac{a-b}{\sqrt{(a-b)^2+4\mu^2}} \\ -\frac{4\mu}{\sqrt{(a-b)^2+4\mu^2}} \end{pmatrix}, \tag{2.15}$$

and

$$\nabla^2\phi(a,b,\mu) = \frac{4}{((a-b)^2+4\mu^2)^{\frac{3}{2}}} \begin{pmatrix} -\mu^2 & \mu^2 & (a-b)\mu \\ \mu^2 & -\mu^2 & -(a-b)\mu \\ (a-b)\mu & -(a-b)\mu & -(a-b)^2 \end{pmatrix}. \tag{2.16}$$

Since $\nabla^2\psi(a,b,\mu)$ is negative semi-definite for $(a,b,\mu) \in I\!R^2 \times R_+$, the function $\phi(a,b,\mu)$ is concave on $I\!R^2 \times I\!R_+$. □

3 A Predictor-Corrector Algorithm

In this section, we state our predictor-corrector algorithm and show that it is well–defined.

The Algorithm

Step 0: (Initialization)

Choose $x^0 \in I\!R^n$, set $y^0 = Mx^0 + q$, and let $\mu_0 > 0$ be such that $\Phi(x^0, y^0, \mu_0) < 0$. Choose $\beta > 2\sqrt{n}$ so that $\|\Phi(x^0, y^0, \mu_0)\| \le \beta\mu_0$. We now have $(x^0, y^0) \in \mathcal{N}(\beta, \mu_0)$. Choose $\bar{\sigma}$, α_1, and α_2 from $(0,1)$.

Step 1: (The Predictor Step)

Let $(\Delta x^k, \Delta x^k, \Delta\mu_k)$ solve the equation

$$F(x^k, y^k, \mu_k) + \nabla F(x^k, y^k, \mu_k)^T \begin{bmatrix} \Delta x^k \\ \Delta x^k \\ \Delta\mu_k \end{bmatrix} = 0 \tag{3.1}$$

If $\left\|\Phi(x^k + \Delta x^k, y^k + \Delta y^k, 0)\right\| = 0$, **STOP**, $(x^k + \Delta x^k, y^k + \Delta y^k)$ solves LCP(q, M) ; else if

$$\left\|\Phi(x^k + \Delta x^k, y^k + \Delta y^k, \mu_k)\right\| > \beta\mu_k,$$

set

$$\hat{x}^k := x^k, \quad \hat{y}^k := y^k, \quad \hat{\mu}_k := \mu_k, \quad \text{and} \quad \eta_k = 1 \; ; \tag{3.2}$$

else let $\eta_k = \alpha_1^s$ where s is the positive integer such that

$$\left\|\Phi(x^k + \Delta x^k, y^k + \Delta y^k, \alpha_1^t \mu_k)\right\| \;\leq\; \alpha_1^t \beta \mu_k, \quad \text{for} \quad t = 0, 1, 2, \ldots, s, \quad \text{and} \tag{3.3}$$

$$\left\|\Phi(x^k + \Delta x^k, y^k + \Delta y^k, \alpha_1^{s+1} \mu_k)\right\| \;>\; \alpha_1^{s+1} \beta \mu_k. \tag{3.4}$$

Set

$$\hat{x}^k := x^k + \Delta x^k, \quad \hat{y}^k := y^k + \Delta y^k, \quad \hat{\mu}_k := \eta_k \mu_k. \tag{3.5}$$

Step 2: (The Corrector Step)

Let $(\Delta\hat{x}^k, \Delta\hat{y}^k, \Delta\hat{\mu}_k)$ solve the equation

$$F(\hat{x}^k, \hat{y}^k, \hat{\mu}_k) + \nabla F(\hat{x}^k, \hat{y}^k, \hat{\mu}_k)^T \begin{bmatrix} \Delta\hat{x}^k \\ \Delta\hat{x}^k \\ \Delta\hat{\mu}_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (1 - \bar{\sigma})\hat{\mu}_k \end{bmatrix} \tag{3.6}$$

and let $\hat{\lambda}_k$ be the maximum of the value $1, \alpha_2, \alpha_2^2, \ldots,$ such that

$$\left\|\Phi(\hat{x}^k + \hat{\lambda}_k \Delta\hat{x}^k, \hat{y}^k + \hat{\lambda}_k \Delta\hat{y}^k, (1 - \bar{\sigma}\hat{\lambda}_k)\hat{\mu}_k)\right\| \leq (1 - \bar{\sigma}\hat{\lambda}_k)\beta\hat{\mu}_k. \tag{3.7}$$

Set

$$x^{k+1} = \hat{x}^k + \hat{\lambda}_k \Delta\hat{x}^k, \quad y^{k+1} = \hat{y}^k + \hat{\lambda}_k \Delta\hat{y}^k, \quad \mu_{k+1} = (1 - \bar{\sigma}\hat{\lambda}_k)\hat{\mu}_k, \tag{3.8}$$

and return to Step 1.

Remarks 1. Note that if the null step (3.2) is taken in Step 1, then the Newton equations (3.1) and (3.6) have the same coefficient matrix. Therefore only one matrix factorization is needed to implement both Steps 1 and 2.

2. In the initialization step, setting

$$\mu_0 > \sqrt{\max_{\substack{i \in \{1,\ldots,n\} \\ 0 < x_i^0, \; 0 < y_i^0}} \{0, x_i^0 y_i^0\}}$$

guarantees that the inequality $\Phi(x^0, y^0, \mu_0) < 0$ is satisfied. For example, one can choose $(x^0, y^0) = (0, q)$ in which case μ_0 can be taken to be any positive number.

3. The condition that $\beta > 2\sqrt{n}$ is only employed in proof of local quadratic convergence. It is not required to verify the global linear convergence of the method.

4. Observe that the function F has nonsingular Jacobian at a given point if and only if $\nabla_{(x,y)} F$ is nonsingular at that point. In [13, Theorem 3.5], it is shown that if $\mu > 0$ and the matrix M is a P_0 matrix, then $\nabla_{(x,y)} F(\bar{x}, \bar{y}, \mu)$ is nonsingular for all $(\bar{x}, \bar{y}) \in I\!R^{2n}$. Therefore, since the matrix M is assumed to be positive semi–definite and the algorithm is initiated with $\mu_0 > 0$ and terminates if $\mu_k = 0$, the Jacobians $\nabla F(x^k, y^k, \mu_k)$

and $\nabla F(\hat{x}^k, \hat{y}^k, \hat{\mu}_k)$ are always nonsingular and so the Newton equations (3.1) and (3.6) yield unique solutions whenever (x^k, y^k, μ_k) and $(\hat{x}^k, \hat{y}^k, \hat{\mu}_k)$ are well–defined. In addition, since $y^0 = Mx^0 + q$, we have $y^k = Mx^k + q$ and $\hat{y}^k = M\hat{x}^k + q$ for all well–defined iterates.

In analyzing the algorithm, it is helpful to take a closer look at the Newton equations (3.1) and (3.6). In (3.1) we have $\Delta\mu_k = -\mu_k$ and so (3.1) reduces to the system

$$\begin{gathered} M\Delta x^k - \Delta y^k = 0 \\ \nabla_x \Phi(x^k, y^k, u_k)^T \Delta x + \nabla_y \Phi(x^k, y^k, u_k)^T \Delta y = -\Phi(x^k, y^k, \mu_k) + \mu_k \nabla_\mu \Phi(x^k, y^k, u_k). \end{gathered} \tag{3.9}$$

Similarly, in (3.6), $\Delta\hat{\mu}_k = -\bar{\sigma}\hat{\mu}_k$ reducing (3.6) to the system

$$\begin{gathered} M\Delta \hat{x}^k - \Delta \hat{y}^k = 0 \\ \nabla_x \Phi(\hat{x}^k, \hat{y}^k, \hat{u}_k)^T \Delta \hat{x} + \nabla_y \Phi(\hat{x}^k, \hat{y}^k, \hat{u}_k)^T \Delta \hat{y} = -\Phi(\hat{x}^k, \hat{y}^k, \hat{\mu}_k) + \bar{\sigma}\hat{\mu}_k \nabla_\mu \Phi(\hat{x}^k, \hat{y}^k, \hat{u}_k). \end{gathered} \tag{3.10}$$

Theorem 3.1 *Consider the algorithm described above and suppose that the matrix M is a P_0 matrix. If $(x^k, y^k) \in \mathcal{N}(\beta, \mu_k)$ with $\mu_k > 0$, then either $(x^k + \Delta x^k, y^k + \Delta y^k)$ solves LCP(q, M) or both $(\hat{x}^k, \hat{y}^k, \hat{\mu}_k)$ and $(x^{k+1}, y^{k+1}, \mu_{k+1})$ are well–defined with the backtracking routines in Steps 1 and 2 finitely terminating. In the latter case, we have $(\hat{x}^k, \hat{y}^k) \in \mathcal{N}(\beta, \hat{\mu}_k)$ and $(x^{k+1}, y^{k+1}) \in \mathcal{N}(\beta, \mu_{k+1})$ with $0 < \mu_{k+1} < \hat{\mu}_k$. Since $(x^0, y^0) \in \mathcal{N}(\beta, \mu_0)$ with $\mu_0 > 0$, this shows that the algorithm is well–defined.*

Proof Let $(x^k, y^k) \in \mathcal{N}(\beta, \mu_k)$ with $\mu_k > 0$. By Remark 4 above, $(\Delta x^k, \Delta y^k, \Delta\mu_k)$ exists and is unique. Since $y^k + \Delta y^k = M(x^k + \Delta x^k) + q$, we have $\left\|\Phi(x^k + \Delta x^k, y^k + \Delta y^k, 0)\right\| = 0$ if and only if $(x^k + \Delta x^k, y^k + \Delta y^k)$ solves LCP(q, M). Therefore, if $(x^k + \Delta x^k, y^k + \Delta y^k)$ does not solve LCP(q, M), then by continuity, there exist $\epsilon > 0$ and $\bar{\mu} > 0$ such that $\left\|\Phi(x^k + \Delta x^k, y^k + \Delta y^k, \mu)\right\| > \epsilon$ for all $\mu \in [0, \bar{\mu}]$. In this case, the backtracking routine described in (3.3) and (3.4) of Step 2 is finitely terminating. Hence $(\hat{x}^k, \hat{y}^k, \hat{\mu}_k)$ is well–defined, with $0 < \hat{\mu}_k \leq \mu_k$, and $(\Delta\hat{x}^k, \Delta\hat{y}^k, \Delta\hat{\mu}_k)$ is uniquely determined by (3.6). To see that the backtracking routine in Step 3 is finitely terminating, define $\psi(x, y, \mu) = \|\Phi(x, y, \mu)\|$ and note that by (3.6)

$$\psi'((\hat{x}^k, \hat{y}^k, \hat{\mu}_k); (\Delta\hat{x}^k, \Delta\hat{y}^k, \Delta\hat{\mu}_k)) = -\left\|\Phi(\hat{x}^k, \hat{y}^k, \hat{\mu}_k)\right\| .$$

Therefore, (3.7) can be viewed as an instance of a standard backtracking line search routine and as such is finitely terminating with $0 < \mu_{k+1} < \hat{\mu}_k \leq \mu_k$ (indeed, one can replace the value of $\bar{\sigma}$ on the right hand side of (3.7) by any number in the open interval $(0, 1)$).

Since $(x^k, y^k) \in \mathcal{N}(\beta, \mu_k)$, the argument given above implies that either $(x^k + \Delta x^k, y^k + \Delta y^k)$ solves LCP(q, M) or $\hat{y}^k = M\hat{x}^k + q$ with $\left\|\Phi(\hat{x}^k, \hat{y}^k, \hat{\mu}_k)\right\| \leq \beta\hat{\mu}_k$ and $y^{k+1} = Mx^{k+1} + q$ with $\left\|\Phi(x^{k+1}, y^{k+1}, \mu_{k+1})\right\| \leq \beta\mu_{k+1}$. Thus, if $(x^k + \Delta x^k, y^k + \Delta y^k)$ does not solve LCP(q, M), we need only show that $\Phi(\hat{x}^k, \hat{y}^k, \hat{\mu}_k) \leq 0$ and $\Phi(x^{k+1}, y^{k+1}, \mu_{k+1}) \leq 0$ in order to have $(\hat{x}^k, \hat{y}^k) \in \mathcal{N}(\beta, \hat{\mu}_k)$ and $(x^{k+1}, y^{k+1}) \in \mathcal{N}(\beta, \mu_{k+1})$. First note that the componentwise concavity of Φ implies that for any $(x, y, \mu) \in \mathbb{R}^{2n+1}$ with $\mu > 0$, and $(\Delta x, \Delta y, \Delta \mu) \in \mathbb{R}^{2n+1}$ one has

$$\Phi(x + \Delta x, y + \Delta y, \mu + \Delta\mu) \leq \Phi(x, y, \mu) + \nabla\Phi(x, y, \mu)^T \begin{bmatrix} \Delta x \\ \Delta x \\ \Delta\mu \end{bmatrix}.$$

Hence, in the case of the predictor step, either (3.2) holds or

$$\begin{aligned}
& \Phi(\hat{x}^k, \hat{y}^k, \hat{\mu}_k) \\
= \; & \Phi(x^k + \Delta x^k, y^k + \Delta y^k, \eta_k \mu_k) \\
\leq \; & \Phi(x^k, y^k, \mu_k) + \nabla\Phi(x^k, y^k, \mu_k)^T \begin{pmatrix} \Delta x^k \\ \Delta y^k \\ (\eta_k - 1)\mu_k \end{pmatrix} \\
= \; & \Phi(x^k, y^k, \mu_k) + \nabla\Phi(x^k, y^k, \mu_k)^T \begin{pmatrix} \Delta x^k \\ \Delta y^k \\ -\mu_k \end{pmatrix} + \eta_k \mu_k \nabla_\mu \Phi(x^k, y^k, \mu_k) \\
= \; & \eta_k \mu_k \nabla_\mu \Phi(x^k, y^k, \mu_k) \\
= \; & \eta_k \mu_k \left(\frac{-4\mu_k}{\sqrt{(x^k - y^k)^2 + 4\mu_k^2}} \right) \leq 0.
\end{aligned}$$

In either case, $\Phi(\hat{x}^k, \hat{y}^k, \hat{\mu}_k) \leq 0$. For the corrector step, we have

$$\begin{aligned}
& \Phi(x^{k+1}, y^{k+1}, \mu_{k+1}) \\
= \; & \Phi(\hat{x}^k + \hat{\lambda}_k \Delta\hat{x}^k, \hat{y}^k + \hat{\lambda}_k \Delta\hat{y}^k, \hat{\mu}_k + \hat{\lambda}_k \Delta\hat{\mu}_k) \\
\leq \; & \Phi(\hat{x}^k, \hat{y}^k, \hat{\mu}_k) + \hat{\lambda}_k \nabla\Phi(\hat{x}^k, \hat{y}^k, \hat{\mu}_k)^T \begin{bmatrix} \Delta\hat{x}^k \\ \Delta\hat{y}^k \\ -\bar{\sigma}\hat{\mu}_k \end{bmatrix} \\
= \; & (1 - \hat{\lambda}_k)\Phi(\hat{x}^k, \hat{y}^k, \hat{\mu}_k) \leq 0,
\end{aligned}$$

since we have already shown that $\Phi(\hat{x}^k, \hat{y}^k, \hat{\mu}_k) \leq 0$. This completes the proof. □

4 Convergence

We require the following key assumption:

Assumption (B): Given $\beta > 0$ and $\mu_0 > 0$, there exists a $C > 0$ such that

$$\left\| \nabla_{(x,y)} F(\bar{x}, \bar{y}, \mu)^{-1} \right\| \leq C, \tag{4.1}$$

for all $0 < \mu \leq \mu_0$ and $(\bar{x}, \bar{y}) \in \mathcal{N}(\beta, \mu)$.

In [1, Proposition 4.3], we show that such a bound exits under the assumption of a non–degeneracy condition due to Fukushima, Luo, and Pang [9, Assumption (A2)]. Similar results of this type have since been obtained by Chen and Xiu [5, Section 6], Tseng [23, Corollary 2], and Qi and Sun [18, Proposition 2]. In private discussions, Kanzow [14] points out that the Fukushima, Luo, and Pang non–degeneracy condition implies the uniqueness of the solution to LCP(q, M). Kanzow's proof, which we give below, easily extends to show that *any* condition which implies Assumption (B) also implies the uniqueness of the solution to LCP(q, M).

Proposition 4.1 *[14] If Assumptions (A) and (B) hold, then LCP(q, M) has a unique solution.*

Proof Let S denote the set of solutions to LCP(q, M). Assumption (A) guarantees S is non–empty. Assumption (B) implies the non–singularity of every element of the so–called B-subdifferential [17, page 233] of F at every point in S. Thus, by [17, Proposition 2.5], S contains only isolated points. But then, by [12, Corollary 5], S must be a singleton. □

We are now ready to establish the global linear convergence of the algorithm. This result depends only on the corrector step (Step 2 of the algorithm) and is independent of whether or not the predictor step (Step 1 of the algorithm) is implemented on any given iteration. It is also interesting to note that this approach to global linear convergence is considerably simpler than the approach given in [1] since the update to the iterates (x^k, y^k) and the smoothing parameter μ_k are computed simultaneously.

Theorem 4.2 *(Global Linear Convergence) Suppose that Assumptions (A) and (B) hold. Let $\{(x^k, y^k, \mu_k)\}$ be the sequence generated by the algorithm. If the algorithm does not terminate finitely at the unique solution to LCP(q, M), then for $k = 0, 1, \ldots,$*

$$(x^k, y^k) \in \mathcal{N}(\beta, \mu_k) \tag{4.2}$$

$$(1 - \bar{\sigma}\hat{\lambda}_{k-1})\eta_{k-1} \ldots (1 - \bar{\sigma}\hat{\gamma}_0)\eta_0\mu_0 \quad = \quad \mu_k, \tag{4.3}$$

with

$$\hat{\lambda}_k \geq \bar{\lambda} := \min\{1, \frac{\alpha_2(1-\bar{\sigma})\beta}{C^2(\beta + 2\bar{\sigma})^2 + \sqrt{n}\bar{\sigma}^2 + \bar{\sigma}(1-\bar{\sigma})\beta}\}, \tag{4.4}$$

where C is the constant defined in (4.1). Therefore μ_k converges to 0 at a global linear rate. In addition, the sequence $\{(x^k, y^k)\}$ converges to the unique solution of LCP(q, M).

Proof The inclusion (4.2) has already been established in Theorem 3.1 and the relation (4.3) follows by construction.

For sake of simplicity, set $(x, y, \mu) = (\hat{x}_k, \hat{y}_k, \hat{\mu}_k)$ and $(\Delta x, \Delta y) = (\Delta\hat{x}^k, \Delta\hat{y}^k)$. Then for $i \in \{1, \ldots, n\}$ and $\lambda \in [0, 1]$, Lemma 2.2 and (3.6) imply that

$$\begin{aligned} & |\phi(x_i + \lambda\Delta x_i, y_i + \lambda\Delta y_i, (1 - \bar{\sigma}\lambda)\mu)| \\ = & \left| \phi(x_i, y_i, \mu) + \lambda\nabla\phi(x_i, y_i, \mu)^T \begin{pmatrix} \Delta x_i \\ \Delta y_i \\ -\bar{\sigma}\mu_k \end{pmatrix} + \right. \end{aligned}$$

$$\left|\frac{\lambda^2}{2}\begin{pmatrix}\Delta x_i\\ \Delta y_i\\ -\bar\sigma\mu_k\end{pmatrix}^T \nabla^2\phi(x_i+\theta_i\lambda\Delta x_i, y_i+\theta_i\lambda\Delta y_i,(1-\theta_i\bar\sigma\lambda)\mu)\begin{pmatrix}\Delta x_i\\ \Delta y_i\\ -\bar\sigma\mu_k\end{pmatrix}\right|$$

$$\begin{aligned}
&\le (1-\lambda)|\phi(x_i^k,y_i^k,\mu)| + \frac{\lambda^2}{2}\left\|\nabla^2\phi(x_i+\theta_i\lambda\Delta x_i, y_i+\theta_i\lambda\Delta y_i,(1-\theta_i\bar\sigma\lambda)\mu)\right\|\left\|\begin{pmatrix}\Delta x_i\\ \Delta y_i\\ -\bar\sigma\mu_k\end{pmatrix}\right\|^2\\
&\le (1-\lambda)|\phi(x_i,y_i,\mu)| + \frac{\lambda^2}{(1-\bar\sigma\lambda)\mu}\|(\Delta x_i,\Delta y_i,-\bar\sigma\mu)\|^2
\end{aligned}$$

for some $\theta_i\in[0,1]$. Set $t_i := \|(\Delta x_i,\Delta y_i,-\bar\sigma\mu_k)\|^2$ for $i=1,\ldots,n$, then

$$\begin{aligned}
&\|\Phi(x+\lambda\Delta x, y+\lambda\Delta y,(1-\bar\sigma\lambda)\mu)\|\\
\le\ & (1-\lambda)\|\Phi(x,y,\mu)\| + \frac{\lambda^2}{(1-\bar\sigma\lambda)\mu}\left\|\begin{pmatrix}t_1\\ \cdots\\ t_n\end{pmatrix}\right\|\\
\le\ & (1-\lambda)\|\Phi(x,y,\mu)\| + \frac{\lambda^2}{(1-\bar\sigma\lambda)\mu}\left(\left\|\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}\right\|^2 + \sqrt{n}\bar\sigma^2\mu^2\right)\\
\le\ & (1-\lambda)\beta\mu + \frac{\lambda^2}{1-\bar\sigma\lambda}\left(C^2(\beta+2\bar\sigma)^2+\sqrt{n}\bar\sigma^2\right)\mu, \qquad (4.5)
\end{aligned}$$

where the last inequality follows from (3.9) which yields the bound

$$\left\|\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}\right\| \le \left\|\nabla_{(x,y)}F^{-1}(x,y,\mu)\right\|(\|\Phi(x,y,\mu)\| + \bar\sigma\mu\|\nabla_\mu\Phi(x,y,\mu)\|) \le C(\beta+2\bar\sigma)\mu. \quad (4.6)$$

It is easily verified that

$$(1-\lambda)\beta\mu + \frac{\lambda^2}{1-\bar\sigma\lambda}[C^2(\beta+2\bar\sigma)^2+\sqrt{n}\bar\sigma^2]\mu \le (1-\bar\sigma\lambda)\beta\mu,$$

whenever

$$\lambda \le \frac{(1-\bar\sigma)\beta}{C^2(\beta+2\bar\sigma)^2+\sqrt{n}\bar\sigma^2+\bar\sigma(1-\bar\sigma)\beta}.$$

Therefore

$$\hat\lambda_k \ge \min\left\{1, \frac{\alpha_2(1-\bar\sigma)\beta}{C^2(\beta+2\bar\sigma)^2+\sqrt{n}\bar\sigma^2+\bar\sigma(1-\bar\sigma)\beta}\right\}.$$

To conclude, note that the sequence $\{(x^k,y^k)\}$ is bounded by Lemma 2.1 and Theorem 3.1. In addition, just as in (4.6), the relations (3.9) and (3.10) yield the bounds

$$\left\|\begin{pmatrix}\Delta x^k\\ \Delta y^k\end{pmatrix}\right\| \le C(\beta+2)\mu_k \quad\text{and}\quad \left\|\begin{pmatrix}\Delta\hat x^k\\ \Delta\hat y^k\end{pmatrix}\right\| \le C(\beta+2)\mu_k\ ,$$

since $0<\bar\sigma<1$ and $0<\eta_k\le 1$ for all k. Therefore, (4.3) and (4.4) imply that

$$\begin{aligned}
\left\|(x^{k+1},y^{k+1})-(x^k,y^k)\right\| &\le \left\|\begin{pmatrix}\Delta x^k\\ \Delta y^k\end{pmatrix}\right\| + \hat\lambda_k\left\|\begin{pmatrix}\Delta\hat x^k\\ \Delta\hat y^k\end{pmatrix}\right\|\\
&\le 2C(\beta+2)\mu_k \le 2C(\beta+2)(1-\bar\sigma\bar\lambda)^k\mu_0\ .
\end{aligned}$$

Hence, $\{(x^k, y^k)\}$ is a Cauchy sequence and so must converge to the unique solution of LCP(q, M). $\square$

We now establish the local quadratic convergence of the μ_k's under the assumption that the iterates converge to a solution of LCP(q, M) at which strict complementary slackness is satisfied.

Theorem 4.3 *(Local Quadratic Convergence) Suppose Assumption (B) holds and that the sequence $\{(x^k, y^k, \mu_k)\}$ generated by the algorithm converges to $\{(x^*, y^*, 0)\}$ where $\{(x^*, y^*)\}$ is the unique solution to LCP(q, M). If it is further assumed that the strict complementary slackness condition $0 < x^* + y^*$ is satisfied, then*

$$\mu_{k+1} = O(\mu_k^2), \tag{4.7}$$

that is, μ_k converges quadratically to zero.

Proof First observe that due to the strict complementarity of $\{(x^*, y^*)\}$, Part (iii) of Lemma 2.2 indicates that there exist constants $\epsilon > 0$ and $L > 0$ such that

$$\left\|\nabla^2 \phi(x, y, \mu)\right\| \le L, \quad \text{whenever} \quad \|(x, y, \mu) - (x^*, y^*, 0)\| \le \epsilon. \tag{4.8}$$

Hence, for all k sufficient large and $\eta \in (0, 1]$, we have for each $i \in \{1, \dots, n\}$ that

$$\begin{aligned}
& |\phi(x_i^k + \Delta x_i^k, y_i^k + \Delta y_i^k, \eta\mu_k)| \\
= & \left|\phi(x_i^k, y_i^k, \mu_k) + \nabla^T \phi(x_i^k, y_i^k, \mu_k) \begin{pmatrix} \Delta x_i^k \\ \Delta y_i^k \\ (\eta - 1)\mu_k \end{pmatrix}\right| + \\
& \frac{1}{2}\left| \begin{pmatrix} \Delta x_i^k \\ \Delta y_i^k \\ (\eta - 1)\mu_k \end{pmatrix}^T \nabla^2 \phi(x_i^k + \theta_i \Delta x_i^k, y_i + \theta_i \Delta y_i^k, (1 + \theta_i(\eta - 1))\mu_k) \begin{pmatrix} \Delta x_i^k \\ \Delta y_i^k \\ (\eta - 1)\mu_k \end{pmatrix}\right| \\
= & \left|\phi(x_i^k, y_i^k, \mu_k) + \nabla^T \phi(x_i^k, y_i^k, \mu_k) \begin{pmatrix} \Delta x_i^k \\ \Delta y_i^k \\ -\mu_k \end{pmatrix} + \eta\mu_k \nabla_\mu \phi(x^k, y^k, \mu_k)\right| + \\
& \frac{1}{2}\left| \begin{pmatrix} \Delta x_i^k \\ \Delta y_i^k \\ (\eta - 1)\mu_k \end{pmatrix}^T \nabla^2 \phi(x_i^k + \theta_i \Delta x_i^k, y_i + \theta_i \Delta y_i^k, (1 + \theta_i(\eta - 1))\mu) \begin{pmatrix} \Delta x_i^k \\ \Delta y_i^k \\ (\eta - 1)\mu_k \end{pmatrix}\right| \\
\le & \ \eta\mu_k |\nabla_\mu \phi(x^k, y^k, \mu_k)| + \frac{L}{2}\left\|(\Delta x_i^k, \Delta y_i^k, (\eta - 1)\mu_k)\right\|^2 \\
= & \ 2\eta\mu_k + \frac{L}{2}\left(\left\|(\Delta x_i^k, \Delta y_i^k)\right\|^2 + (1 - \eta)^2 \mu_k^2\right)
\end{aligned}$$

Now using an argument similar to that used to obtain (4.5), we have

$$\begin{aligned}
\left\|\Phi(x^k + \Delta x^k, y^k + \Delta y^k, \eta\mu_k)\right\| & \le 2\sqrt{n}\ \eta\mu_k + \frac{L}{2}(C^2(\beta + 2)^2 + \sqrt{n}(1 - \eta)^2)\mu_k^2 \\
& \le 2\sqrt{n}\ \eta\mu_k + \frac{L}{2}(C^2(\beta + 2)^2 + \sqrt{n})\mu_k^2.
\end{aligned} \tag{4.9}$$

Hence, since $\beta > 2\sqrt{n}$, the inequality (3.3) in Step 1 of the algorithm holds with $t = 0$ for all k sufficiently large. It is easy to verify that

$$2\sqrt{n}\ \eta\mu_k + \frac{L}{2}(C^2(\beta+2)^2 + \sqrt{n})\mu_k^2 \le \eta\beta\mu_k, \tag{4.10}$$

whenever

$$\eta \ge \frac{L}{2(\beta - 2\sqrt{n})}[C^2(\beta+2)^2 + \sqrt{n}]\mu_k.$$

Hence, by (3.4), we have

$$\alpha_1\eta_k \le \frac{L}{2(\beta - 2\sqrt{n})}[C^2(\beta+2)^2 + \sqrt{n}]\mu_k,$$

and so

$$\eta_k \le \frac{L}{2\alpha_1(\beta - 2\sqrt{n})}[C^2(\beta+2)^2 + \sqrt{n}]\mu_k, \tag{4.11}$$

for all k sufficient large. Therefore, by (3.5),

$$\mu_{k+1} = O(\mu_k^2).$$

□

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