On the Clarke Subdifferential
of the Distance Function of a Closed Set

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Let C be a nonempty closed subset of the real normed linear space X. In this paper we determine the extent to which formulas for the Clarke subdifferential of the distance for C,

$$d_C(x) := \inf_{y \in C} \|x - y\|,$$

which are valid when C is convex, remain valid when C is not convex. The assumption of subdifferential regularity for d_C plays an important role. When $x \notin C$, the most precise results also require the nonemptiness of the set

$$P_C(x) := \{\bar{x} \in C : \|x - \bar{x}\| = d_C(x)\}.$$

As an interesting side result, the equivalence between the strict differentiability and the regularity of d_C at x is established when $x \notin C$, $P_C(x) \neq \varnothing$, and the norm on X is smooth.   © 1992 Academic Press, Inc.

1. INTRODUCTION

Let C be a nonempty closed subset of the normed linear space X. We consider various formulas for the subdifferential of the distance function

$$d_C(x) := \inf_{y \in C} \|x - y\|.$$

Such formulas are fundamental to many applications, e.g., multiplier existence theorems in constrained optimization [3], algorithms for solving nonlinear systems of equations, and nonlinear programs (e.g., see [2] and [4]). If the set C is convex, then d_C is a convex function. In this case, a great deal is known about the subdifferential of d_C. We review these results in Section 2. When C is not convex, we determine to what extent the formulas for the convex case remain valid. In this regard, the subdifferential regularity of the function d_C plays a key role in the analysis. Under this assumption, the Clarke subdifferential of d_C behaves in a fashion similar to the convex case. The cases $x \in C$ and $x \notin C$ are examined separately in Sections 3 and 4 of the paper, respectively.

The characterization of the points of differentiability of d_C is a topic to which a great deal of effort has been devoted (e.g., see [1, 7, 10]). This work, especially that of Borwein, Fitzpatrick, and Giles [1], is closely related to our own. However, our objective is distinctly different, since we are interested in formulas for the subdifferential regardless of differentiability. In particular, we are interested in those formulas that exhibit a geometry similar to that of the convex case.

The notation that we employ is for the most part standard; however, a partial list is provided for the reader's convenience. Let X be real normed linear space and let X^* be its topological dual. The spaces X and X^* are paired in duality by the continuous bi-linear form

$$\langle x^*, x \rangle := x^*(x)$$

defined on $X^* \times X$. We denote the norms on X and X^* by $\|\cdot\|$ and $\|\cdot\|_0$, respectively. The associated closed unit balls are denoted by $\mathbb{B}$ and $\mathbb{B}^0$, respectively. Given two subsets A and B of X (or X^*) and $\beta \in \mathbb{R}$, we define

$$A \pm \beta B := \{a \pm \beta b : a \in A, b \in B\}.$$

On the other hand,

$$A \backslash B := \{a \in A : a \notin B\}.$$

If $A \subset X$, then the polar of A is defined to be the set

$$A^0 := \{x^* \in X^* : \langle x^*, x \rangle \leqslant 1 \; \forall x \in A\}.$$

The indicator function for A is given by

$$\psi_A(x) := \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise.} \end{cases}$$

If $B \subset X^*$, then the support function associated with B is given by

$$\psi_B^*(x) := \sup\{\langle x^*, x \rangle : x^* \in B\}.$$

Let $f: X \mapsto \mathbb{R}$. Then the directional derivative of f at a point $x \in X$ in the direction $d \in X$ is given by

$$f'(x; d) := \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}$$

whenever this limit exists. The contingent directional derivative of f at x in the direction d is given by

$$f^-(x; d) := \liminf_{\substack{u \to d \\ t \downarrow 0}} \frac{f(x + tu) - f(x)}{t}.$$

Let us now assume that f is locally Lipschitzian. Then the Clarke directional derivative of f at x in the direction d is given by

$$f^0(x; d) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}.$$

The Clarke subdifferential of f at x is given by

$$\partial f(x) := \{x^* \in X^* : \langle x^*, d \rangle \leqslant f^0(x; d) \ \forall d \in X\}.$$

Consequently, we have

$$f^0(x; \cdot) = \psi_{\partial f(x)}^*(\cdot).$$

Moreover, if L is a local Lipschitz constant for f at x, then $\partial f(x) \subset L\mathbb{B}^0$ and $f^0(x; d) \leqslant L \|d\|$ for all $d \in X$. We say that f is strictly differentiable at a point $x \in X$ if there exists $v \in X^*$ such that

$$\langle v, d \rangle = \lim_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}$$

for all $d \in X$. The vector v is called the strict derivative of f at x and is denoted $f'(x)$.

The distance function for a set $A \in X$ is Lipschitz with global Lipschitz constant 1. Therefore its Clarke directional derivative exists and is finite at every point in all directions. Based on this observation one defines the tangent cone at a point $x \in A$ by

$$T_A(x) := \{d \in X : d^0_C(x; d) \leqslant 0\}.$$

The polar of the tangent cone is called the normal cone and is denoted by

$$N_A(x) := T_A(x)^0.$$

Finally, the contingent cone to A at a point $x \in A$ is given by

$$K_A(x) := \{d \in X : \exists t_k \downarrow 0, d^k \to d, \text{ with } x + t_k d^k \in A\}.$$

2. THE CONVEX CASE

In this section we assume that the set C is nonempty, closed, and convex. In this case the subdifferential properties of d_C can be derived from the observation that d_C is the infimal convolution of the norm and the indicator function for C (see [9]).

THEOREM 1.   *Let C be a non-empty, closed, and convex subset of X. Then d_C is a convex function on X with convex subdifferential*

$$\partial d_C(x) = \begin{cases} \text{bdry}(\mathbb{B}^0) \cap N_{C_x}(x) & \text{if } x \notin C \\ \mathbb{B}^0 \cap N_C(x) & \text{if } x \in C, \end{cases} \tag{1}$$

where C_x is used to denote the set $C + d_C(x)\mathbb{B}$. If it is further assumed that X is a reflexive Banach space, then the set of points

$$P_C(x) := \{\bar{x} \in C : d_C(x) = \|x - \bar{x}\|\}$$

is nonempty and for any $\bar{x} \in P_C(x)$ one has the formulas

$$\partial d_C(x) = \partial \|x - \bar{x}\| \cap N_{C_x}(x) \tag{2}$$

$$= \partial \|x - \bar{x}\| \cap N_C(\bar{x}). \tag{3}$$

Proof.   For the case in which $x \in C$ formula (1) is well known. The other half of (1) is established in [3, Sect. 2]. The fact that the set $P_C(x)$ is non-empty when X is reflexive is classical. Formula (3) is an immediate consequence of [5, Lemma 3.3]. From the definition of $\partial \|\cdot\|$ and formula (1), we know that the right hand side of formula (2) is contained in $\partial d_C(x)$. Thus, we need only show that the right hand side of (3) is contained in the

right hand side of (2). Indeed, for every $w \in C$, $z \in \mathbb{B}$, and $u \in \partial \|x - \bar{x}\| \cap N_C(\bar{x})$, we have

$$\langle u, w + d_C(x)z - x \rangle = \langle u, \bar{x} - x \rangle + \langle u, w - \bar{x} \rangle + d_C(x)\langle u, z \rangle$$

$$\leqslant -d_C(x) + d_C(x)\langle u, z \rangle + \langle u, w - \bar{x} \rangle$$

$$\leqslant \langle u, w - \bar{x} \rangle$$

$$\leqslant 0,$$

whereby the inclusion follows and the result is established. ∎

COROLLARY 2. *Let C and X be as in Theorem 1. Then $d'_C(x; h)$ exists for all x and h in X and*

$$d'_C(x; h) \leqslant d_{T_{C_x(x)}}(h). \tag{4}$$

Moreover, if $x \in C$, then equality holds,

$$d'_C(x; h) = d_{T_C(x)}(h). \tag{5}$$

Proof. We recall from [4, Theorem 3.1] that if $K \subset X$ is a nonempty, closed, and convex cone, then

$$d_K(x) = \psi^*_{K^0 \cap \mathbb{B}^0}(x).$$

The result now follows from formula (1) and the fact that

$$d'_C(x; h) = \psi^*_{\partial d_C(x)}(h). \quad \blacksquare$$

3. THE NONCONVEX CASE: $x \in C$

In the remainder of the paper we drop the assumption that C is convex and assume only that C is a nonempty closed subset of X. Let us first study the case where $x \in C$. Since d_C is Lipschitz with constant 1, we know from Clarke [6, Proposition 2.1.2] that $\partial d_C(x) \subset \mathbb{B}^0$. Again by Clarke [6, Proposition 2.4.2], we have $\partial d_C(x) \subset N_C(x)$. Therefore it is always the case that

$$\partial d_C(x) \subset \mathbb{B}^0 \cap N_C(x), \tag{6}$$

or equivalently,

$$d^0_C(x; \cdot) \leqslant d_{T_C(x)}(\cdot). \tag{7}$$

However, in general, one cannot complete this inclusion to an equation as in formula (1).

EXAMPLE 3. Let $C \subset \mathbb{R}^2$ be the set $\mathbb{R}^2_+ \cup \mathbb{R}^2_{--}$ under the Euclidean norm. Then it is straightforward to show that $\mathbb{B}^0 \cap N_C(x) = \{(u, v) : u^2 + v^2 \leqslant 1\}$ at $x = (0, 0)$, but $\partial d_C(x) = \{(u, v) : |u| + |v| \leqslant 1\}$.

Although it is not possible to obtain a general formula paralleling formula (1) for the Clarke subdifferential and tangent cone, it is possible, at least in finite dimensions, to obtain a closely related formula for the contingent cone and contingent derivative. Observe that in the convex case, with $x \in C$, formulas (1) and (5) are equivalent. This is not so for non-convex sets C when one uses the contingent cone and the contingent derivative, since neither is assured to be convex. Nonetheless, in the contingent calculus, one can establish (4), and in finite dimensions (5).

THEOREM 4. *If $x \in C$, then*

$$d_C^-(x; h) \leqslant d_{K_C(x)}(h), \tag{8}$$

for all $h \in X$. If it is further assumed that X is finite dimensional, then equality holds in (8).

Remark. It is an open question as to whether or not (8) can be established as an equality in infinite dimensions.

Proof. Choose $h \in X$ and let $v \in K_C(x)$ be such that $\|h - v\| - \varepsilon \leqslant d_{K_C(x)}(h)$. Then there exists $v_n \in t_n^{-1}(C - x)$ with $t_n \downarrow 0$ and $v_n \to v$. Hence

$$d_C^-(x; h) \leqslant \liminf_{n \to \infty} d_{(t_n^{-1}(C-x))}(h)$$

$$\leqslant \lim_{n \to \infty} \|h - v_n\|$$

$$= \|h - v\|$$

$$\leqslant d_{K_C(x)}(h) + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ establishes inequality (8).

Next assume that X is finite dimensional and let (h_n, t_n) be an attaining sequence for the definition of $d_C^-(x; h)$. Let $v_n \in t_n^{-1}(C - x)$ be such that

$$d_{(t_n^{-1}(C-x))}(h_n) \geqslant \|h_n - v_n\| - t_n.$$

Then

$$d_C^-(x; h) = \liminf_{n \to \infty} \frac{d_C(x + t_n h_n) - d_C(x)}{t}$$

$$= \liminf_{n \to \infty} d_{(t_n^{-1}(C-x))}(h_n)$$

$$\geqslant \liminf_{n \to \infty} \|h_n - v_n\|.$$

Since $d_C^-(x; h) < \infty$ by formula (8) and $\{h_n\}$ converges, the sequence $\{v_n\}$ is bounded. Thus some subsequence converges to $v \in K_C(x)$ and so

$$d_C^-(x; h) \geqslant \|h - v\| \geqslant d_{K_C(x)}(h). \quad \blacksquare$$

The following corollary is immediate.

COROLLARY 5.   If $x \in C$, then

$$K_C(x) \subset \{h : d_C^-(x; h) \leqslant 0\}, \tag{9}$$

with equality holding if X is finite dimensional.

In the finite dimensional case it is possible to establish a result paralleling formula (1) for the Clarke subdifferential if one invokes a regularity condition.

DEFINITION 6.   Let $x \in S \subset X$. We say that S is regular at x if

$$T_S(x) = K_S(x).$$

Let $f: X \mapsto \mathbb{R} \cup \{+\infty\}$. We say that f is regular at $x \in X$ if

$$f^0(x; \cdot) = f^-(x; \cdot).$$

PROPOSITION 7.   Let C be a nonempty closed subset of X. If d_C is regular at $x \in C$, then C is regular at x.

Proof.   From inclusion (9) and the definition of $T_C(x)$ we have

$$\begin{aligned} K_C(x) &\subset \{h : d_C^-(x; h) \leqslant 0\} \\ &= \{h : d_C^0(x; h) \leqslant 0\} \\ &\subset T_C(x) \\ &\subset K_C(x). \quad \blacksquare \end{aligned}$$

THEOREM 8.   Let C be a nonempty closed subset of X. If X is finite dimensional and d_C is regular at $x \in C$, then

$$\partial d_C(x) = \mathbb{B}^0 \cap N_C(x) \tag{10}$$

and

$$d_C^0(x; \cdot) = d_{T_C(x)}(\cdot). \tag{11}$$

Proof. By the previous proposition, $K_C(x) = T_C(x)$ and by Theorem 4, $d_C^-(x; \cdot) = d_{K_C(x)}(\cdot)$. This establishes Eq. (11). Equation (10) now follows by convexity. ∎

COROLLARY 9. *Let C be a nonempty closed subset of X and let $x \in C$. If X is finite dimensional, then C is regular at x if and only if d_C is regular at x.*

Proof. The fact that the regularity of d_C implies the regularity of C at x has already been established in Proposition 7. On the other hand if C is regular at x, then, by (7) and Theorem 4,

$$d_C^-(x; h) \leqslant d_C^0(x; h) \leqslant d_{T_C(x)}(h)$$

$$= d_{K_C(x)}(h) = d_C^-(x; h),$$

whereby the result follows. ∎

It is bothersome that we cannot extend Corollary 9 to the infinite dimensional case. The deficiency in our argument resides in the fact that we cannot establish equality in (8) as can be done for finite dimensions. If one could resolve this issue in the affirmative, then the results of this section could be sharpened. However, we are skeptical of the validity of (8) in infinite dimensions.

4. THE NONCONVEX CASE: $x \notin C$

We now concentrate on the case in which $x \notin C$. In this case Borwein, Fitzpatrick, and Giles [1, Theorem 8] establish a very useful characterization of the subdifferential $\partial d_C(x)$ when the norm on X is assumed to be *uniformly Gâteaux differentiable* (see [1, p. 521]). Their characterization provides a strong approximation property for $\partial d_C(x)$. Although our results are somewhat weaker, they do not require the uniform Gâteaux differentiability hypothesis. Many of these results will be derived from the following elementary fact.

LEMMA 10. *Let C be a nonempty closed subset of the normed linear space X. Then for every $x \in X$ and $\alpha \geqslant 0$ one has*

$$d_C(x) \leqslant d_{(C + \alpha \mathbb{B})}(x) + \alpha.$$

Proof. Let $u \in \mathbb{B}$. Then, by the Lipschitz continuity of d_C, we have

$$d_C(x) = d_C((x + \alpha u) - (\alpha u))$$

$$\leqslant d_C(x + \alpha u) + \alpha.$$

Hence

$$d_C(x) \leqslant [\inf_{u \in \mathbb{B}} d_C(x + \alpha u)] + \alpha$$

$$= d_{(C + \alpha \mathbb{B})}(x) + \alpha. \quad \blacksquare$$

By taking $\alpha = d_C(x)$ in the above lemma one obtains the relation

$$d_C(y) \leqslant d_{C_x}(y) + d_C(x). \tag{12}$$

This inequality allows us to extend several of the results of the previous section.

THEOREM 11. *Let C be a nonempty closed subset of X. Then for every $x \in X$, one has*

$$d_C^-(x; \cdot) \leqslant d_{K_{C_x}(x)}(\cdot), \tag{13}$$

$$d_C^0(x; \cdot) \leqslant d_{T_{C_x}(x)}(\cdot), \tag{14}$$

$$K_{C_x}(x) \subset \{h : d_C^-(x; h) \leqslant 0\}, \tag{15}$$

and

$$\partial d_C(x) \subset \mathbb{B}^0 \cap N_{C_x}(x). \tag{16}$$

Proof. When $x \in C$, this result has already been established in the previous section. Regardless, from (12) we always have the inequalities

$$d_C^-(x; \cdot) \leqslant d_{C_x}^-(x; \cdot),$$

and

$$d_C^0(x; \cdot) \leqslant d_{C_x}^0(x; \cdot).$$

Inequalities (13) and (14) are now an immediate consequence of these inequalities and those in (8) and (7). The inclusion (15) follows directly from (13) and inclusion (16) follows from (14) and [4, Theorem 3.1] as in Corollary 2. $\quad \blacksquare$

Remark. One should compare this result with [1, Theorem 8]. Although our inclusion (16) is weaker than this result, we do not restrict the norm on X to be uniformly Gâteaux differentiable.

The remaining results in this section depend upon the nonemptiness of the set

$$P_C(x) := \{\bar{x} \in C : \|x - \bar{x}\| = d_C(x)\}.$$

In the finite dimensional case, $P_C(x)$ is always nonempty as long as C is nonempty and closed. We have the following extension to formula (3) in Theorem 1.

THEOREM 12.   *Let C be a nonempty closed subset of X and let $x \in X$ be such that $P_C(x)$ is nonempty. Then*

$$\partial d_C(x) \subset \mathbb{B}^0 \cap \left(\bigcap_{\bar{y} \in P_C(x)} N_C(\bar{y}) \right) \tag{17}$$

$$\subset \mathbb{B}^0 \cap N_C(\bar{x}) \tag{18}$$

and

$$d_C^0(x; h) \leqslant d_{T_C(x)}(h) \tag{19}$$

for every $\bar{x} \in P_C(x)$.

Proof.   We show that $T_C(\bar{x}) \subset T_{C_x}(x)$. Let $v \in T_C(\bar{x})$ and suppose $x_i \to x$ and $t_i \downarrow 0$. Then $x_i + \bar{x} - x \to \bar{x}$, so that there exists $v_i \to v$ with $x_i + \bar{x} - x + t_i v_i \in C$. It follows that

$$x_i + t_i v_i \in C + (x - \bar{x}) \subset C + d_C(x) \mathbb{B}$$

so that $v \in T_{C_x}(x)$. Thus $d_{T_{C_x}(x)}(h) \leqslant d_{T_C(\bar{x})}(h)$. Formula (19) follows from (14). Formulas (17) and (18) now follow from (19) and [4, Theorem 3.1]. ∎

Formula (1) is now extended by assuming that $P_C(x)$ is nonempty and that d_C is regular at x. We begin with the following preliminary result.

PROPOSITION 13.   *Let C be a nonempty closed subset of X and let $\bar{x} \in P_C(x)$. Then*

$$(1 - t) d_C(x) = d_C(x + t(\bar{x} - x)) \tag{20}$$

for all $t \in [0, 1]$. Moreover, $d_C'(x; \bar{x} - x)$ exists and we have

$$d_C'(x; \bar{x} - x) = d_C^-(x; \bar{x} - x) = -d_C(x). \tag{21}$$

Proof.   To see (20) simply observe that

$$d_C(x) = d_C(x + t(\bar{x} - x) - t(\bar{x} - x))$$
$$\leqslant d_C(x + t(\bar{x} - x)) + t d_C(x),$$

and so $(1 - t) d_C(x) \leqslant d_C(x + t(\bar{x} - x))$. Conversely,

$$d_C(x + t(\bar{x} - x)) = d_C(\bar{x} + (1 - t)(x - \bar{x}))$$

$$\leqslant (1 - t) d_C(x).$$

The fact that $d'_C(x; \bar{x} - x)$ exists and equals $-d_C(x)$ follows immediately from (20). In order to verify the second equality in (21), let $(h_n, t_n) \to (\bar{x} - x, 0)$ be a sequence achieving the limit in the definition of $d_C^-(x; \bar{x} - x)$. Then

$$d_C^-(x; \bar{x} - x) = \lim_n \frac{d_C(x + t_n h_n) - d_C(x)}{t_n}$$

$$\leqslant \lim_n \frac{d_C(x + t_n(\bar{x} - x)) + t_n \|h_n - (\bar{x} - x)\| - d_C(x)}{t_n}$$

$$= \lim_n \frac{(1 - t_n) d_C(x) - d_C(x) + t_n \|h_n - (\bar{x} - x)\|}{t_n}$$

$$= \lim_n - d_C(x) + \|h_n - (\bar{x} - x)\|$$

$$= -d_C(x).$$

The reverse inequality is obtained in exactly the same way, except now one observes that

$$d_C^-(x; \bar{x} - x) \geqslant \lim_n \frac{d_C(x + t_n(\bar{x} - x)) - t_n \|h_n - (\bar{x} - x)\| - d_C(x)}{t_n}. \quad \blacksquare$$

THEOREM 14. Let C be a nonempty closed subset of X and let $x \in X \backslash C$ be such that d_C is regular at x and $P_C(x)$ is nonempty. Then $C_x := C + d_C(x) \mathbb{B}$ is regular at x and for each $\bar{x} \in P_C(x)$ we have

$$\partial d_C(x) = \text{bdry}(\mathbb{B}^0) \cap N_{C_x}(x) \tag{22}$$

$$= \partial \|x - \bar{x}\| \cap N_{C_x}(x) \tag{23}$$

$$= \left(\bigcap_{y \in P_C(x)} \partial \|x - \bar{y}\| \right) \cap N_{C_x}(x) \tag{24}$$

$$\subset \bigcap_{\bar{y} \in P_C(x)} (\partial \|x - \bar{y}\| \cap N_C(\bar{y})) \tag{25}$$

$$\subset \partial \|x - \bar{x}\| \cap N_C(\bar{x}). \tag{26}$$

Proof. The regularity of d_C at x and Proposition 13 imply that $0 \notin \partial d_C(x)$. Consequently, Clarke [6, Theorem 2.4.7] and (15) imply that

$$K_{C_x}(x) \subset \{h : d_C^-(x; h) \le 0\}$$
$$= \{h : d_C^0(x; h) \le 0\}$$
$$\subset T_{C_x}(x).$$

Hence C_x is regular at x. We now show that $\partial d_C(x) \subset \partial \|x - \bar{x}\|$ for any $\bar{x} \in P_C(x)$. Indeed, regularity and (21) imply that $d_C^0(x; \bar{x} - x) = -d_C(x)$, or equivalently,

$$\|x - \bar{x}\| \le \langle z, x - \bar{x} \rangle \qquad \forall z \in \partial d_C(x),$$

for every $\bar{x} \in P_C(x)$. Hence

$$\partial d_C(x) \subset \partial \|x - \bar{x}\| \subset \mathrm{bdry}(\mathbb{B}^0).$$

Therefore, by (16),

$$\partial d_C(x) \subset \partial \|x - \bar{x}\| \cap N_{C_x}(x)$$
$$\subset \mathrm{bdry}(\mathbb{B}^0) \cap N_{C_x}(x).$$

But, by Clarke [6, Corollary 1, p. 56], we have

$$N_{C_x}(x) \subset \bigcup_{\lambda \ge 0} \lambda \partial d_C(x),$$

and so

$$\mathrm{bdry}(\mathbb{B}^0) \cap N_{C_x}(x) \subset \partial d_C(x),$$

whereby (22) and (23) are established. Formula (24) follows immediately from formula (23). Formulas (25) and (26) follow from (23) and (18). ∎

Remark. It is interesting to compare this result with [1, Theorem 5]. In their result Borwein, Fitzpatrick, and Giles show that the Michel–Penot subdifferential [8] of d_C at x always contains an element of $\partial \|x - \bar{x}\|$. Under the regularity hypothesis, the Michel–Penot and Clarke subdifferentials coincide and the stronger statement (23) is possible.

It should be observed that the formulas (22)–(24) are only valid when $0 \notin \partial d_C(x)$. In general, this condition does not hold at points x not in C if d_C is not regular at x.

EXAMPLE 15. Let $X = \mathbb{R}^n$ be given the Euclidean norm and set $C := X \backslash \mathbb{B}$. Then $\partial d_C(0) = \mathbb{B}^0$.

In finite dimensions, recall that if C is regular at $x \in C$, then d_C is regular at x. A similar result for C_x and d_C will not in general be valid when $x \notin C$. This is also illustrated by Example 15.

If X has a smooth norm, Theorem 14 yields an interesting characterization of regularity.

THEOREM 16. *Suppose that the norm on X is smooth. Let C be a non-empty closed subset of X and let $x \in X \backslash C$ be such that $P_C(x) \neq \varnothing$. Then the following statements are equivalent*:

(1) *The function d_C is strictly differentiable at x.*

(2) *The function d_C is regular at x.*

(3) *There exists $\bar{x} \in P_C(x)$ such that*

$$d_C^0(x; \bar{x} - x) = -d_C(x). \tag{27}$$

Furthermore, if any of the above holds, then $P_C(x)$ is a singleton set, and $\partial d_C(x) = \{ (x - \bar{x})/\|x - \bar{x}\| \}$.

Proof. The implication $(1) \Rightarrow (2)$ is a consequence of Clarke [6, Proposition 2.2.4, p. 33]. The implication $(2) \Rightarrow (3)$ follows from Proposition 13. The smoothness of the norm and inclusion (25) imply that $P_C(x)$ is a singleton with $\partial d_C(x) = \{ (x - \bar{x})/\|x - \bar{x}\| \}$.

It remains to show that $(3) \Rightarrow (1)$. To this end, let us suppose that Eq. (27) holds for some $\bar{x} \in P_C(x)$. Then

$$\|x - \bar{x}\| \leqslant \langle z, x - \bar{x} \rangle \qquad \forall z \in \partial d_C(x).$$

Hence

$$\partial d_C(x) \subset \partial \|x - \bar{x}\|.$$

Consequently, $\partial d_C(x)$ is a singleton and so, again by Clarke [6, Proposition 2.2.4, p. 33], d_C is strictly differentiable at x with strict derivative $(x - \bar{x})/\|x - \bar{x}\|$. ∎

The following corollary is an immediate consequence of Theorem 16.

COROLLARY 17. *Let X, C, and x be as in Theorem 16. If $P_C(x)$ contains more that a single element, then d_C is not regular at x.*

5. Concluding Remarks

The goal of this paper was to determine the extent to which the formulas (1), (2), and (3), valid for convex sets C, could be extended to nonconvex sets. For the case in which $x \in C$, formula (1) could only be extended when X is assumed to be finite dimensional and C is regular at x. The problem here is our inability to establish inclusion (9) as an equality in infinite dimensions. Although this inclusion holds as an equality in finite dimensions, it is unknown whether this is so in infinite dimensions. We do not believe that it is. If it is not, then regularity does not imply equality in inclusion (6). This would be somewhat surprising, since then the regularity of C at a point $x \in C$ is not equivalent to the regularity of d_C at x.

When $x \notin C$ we have seen that the basic results can be derived from the case $x \in C$ (Lemma 10 and Theorem 11). However, the full extension of formulas (1) and (2) to the nonconvex case required the nonemptiness of $P_C(x)$ and the regularity of d_C at x (Theorem 14). The proof technique used to establish these results revealed an interesting characterization of the regularity of d_C at x when the norm on X is smooth. That is, d_C is regular at x if and only if d_C is strictly differentiable at x whenever $P_C(x)$ is nonempty (which is always the case in finite dimensions).

The efforts to extend formula (3) were not as successful. The reason for this may be due to the inappropriateness of this formula in the nonconvex case. Indeed, in general, the boundary of the set C_x when $x \notin C$ is better behaved than the boundary of C. In order to give the reader some insight into the problem, we give the following example.

EXAMPLE 18.   Let $C \subset \mathbb{R}^2$ be given by

$$\left\{ (\xi_1, \xi_2) : \xi_2 = \xi_1 \sin\left(\frac{1}{\xi_1}\right), \xi_1 > 0 \right\} \cup \{(0, 0)\}.$$

Then at any point $x = (\xi_1, 0)$ with $\xi_1 < 0$ one has that d_C is regular at x and $P_C(x) = \{\bar{x}\}$ where $\bar{x} = (0, 0)$, but C is not regular at $\bar{x}$.

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