On the Clarke Subdifferential of the Distance Function of a Closed Set

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Submitted by E. Stanley Lee

Received September 4, 1990

Let $C$ be a nonempty closed subset of the real normed linear space $X$. In this paper we determine the extent to which formulas for the Clarke subdifferential of the distance for $C$,

$$d_C(x) := \inf_{y \in C} \|x - y\|,$$

which are valid when $C$ is convex, remain valid when $C$ is not convex. The assumption of subdifferential regularity for $d_C$ plays an important role. When $x \notin C$, the most precise results also require the nonemptiness of the set

$$P_C(x) := \{\bar{x} \in C : \|x - \bar{x}\| = d_C(x)\}.$$

As an interesting side result, the equivalence between the strict differentiability and the regularity of $d_C$ at $x$ is established when $x \notin C$, $P_C(x) \neq \emptyset$, and the norm on $X$ is smooth. © 1992 Academic Press, Inc.

* Supported in part by National Science Foundation Grant DMS-8803206.
† Supported in part by Air Force Office of Scientific Research Grant AFOSR 89-0410.
1. Introduction

Let $C$ be a nonempty closed subset of the normed linear space $X$. We consider various formulas for the subdifferential of the distance function

$$d_C(x) := \inf_{y \in C} \|x - y\|.$$  

Such formulas are fundamental to many applications, e.g., multiplier existence theorems in constrained optimization [3], algorithms for solving nonlinear systems of equations, and nonlinear programs (e.g., see [2] and [4]). If the set $C$ is convex, then $d_C$ is a convex function. In this case, a great deal is known about the subdifferential of $d_C$. We review these results in Section 2. When $C$ is not convex, we determine to what extent the formulas for the convex case remain valid. In this regard, the subdifferential regularity of the function $d_C$ plays a key role in the analysis. Under this assumption, the Clarke subdifferential of $d_C$ behaves in a fashion similar to the convex case. The cases $x \in C$ and $x \notin C$ are examined separately in Sections 3 and 4 of the paper, respectively.

The characterization of the points of differentiability of $d_C$ is a topic to which a great deal of effort has been devoted (e.g., see [1, 7, 10]). This work, especially that of Borwein, Fitzpatrick, and Giles [11], is closely related to our own. However, our objective is distinctly different, since we are interested in formulas for the subdifferential regardless of differentiability. In particular, we are interested in those formulas that exhibit a geometry similar to that of the convex case.

The notation that we employ is for the most part standard; however, a partial list is provided for the reader's convenience. Let $X$ be real normed linear space and let $X^*$ be its topological dual. The spaces $X$ and $X^*$ are paired in duality by the continuous bi-linear form

$$\langle x^*, x \rangle := x^*(x)$$

defined on $X^* \times X$. We denote the norms on $X$ and $X^*$ by $\| \cdot \|$ and $\| \cdot \|_0$, respectively. The associated closed unit balls are denoted by $B$ and $B^0$, respectively. Given two subsets $A$ and $B$ of $X$ (or $X^*$) and $\beta \in \mathbb{R}$, we define

$$A \pm \beta B := \{a \pm \beta b : a \in A, b \in B\}.$$

On the other hand,

$$A \setminus B := \{a \in A : a \notin B\}.$$

If $A \subset X$, then the polar of $A$ is defined to be the set

$$A^0 := \{x^* \in X^* : \langle x^*, x \rangle \leq 1 \ \forall x \in A\}.$$
The indicator function for $A$ is given by

$$\psi_A(x) := \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise.} \end{cases}$$

If $B \subset X^*$, then the support function associated with $B$ is given by

$$\psi_B^*(x) := \sup \{ \langle x^*, x \rangle : x^* \in B \}.$$

Let $f: X \rightarrow \mathbb{R}$. Then the directional derivative of $f$ at a point $x \in X$ in the direction $d \in X$ is given by

$$f'(x; d) := \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}$$
whenever this limit exists. The contingent directional derivative of $f$ at $x$ in the direction $d$ is given by

$$f^-(x; d) := \liminf_{t \downarrow 0} \frac{f(x + tu) - f(x)}{t}. $$

Let us now assume that $f$ is locally Lipschitzian. Then the Clarke directional derivative of $f$ at $x$ in the direction $d$ is given by

$$f^0(x; d) := \limsup_{y \rightarrow x} \frac{f(y + td) - f(y)}{t}.$$ 

The Clarke subdifferential of $f$ at $x$ is given by

$$\partial f(x) := \{ x^* \in X^* : \langle x^*, d \rangle \leq f^0(x; d) \ \forall d \in X \}.$$ 

Consequently, we have

$$f^0(x; \cdot) = \psi_{\partial f(x)}^*(\cdot).$$

Moreover, if $L$ is a local Lipschitz constant for $f$ at $x$, then $\partial f(x) \subset LB^0$ and $f^0(x; d) \leq L \|d\|$ for all $d \in X$. We say that $f$ is strictly differentiable at a point $x \in X$ if there exists $v \in X^*$ such that

$$\langle v, d \rangle = \lim_{y \rightarrow x} \frac{f(y + td) - f(y)}{t}$$
for all $d \in X$. The vector $v$ is called the strict derivative of $f$ at $x$ and is denoted $f'(x)$. 

The distance function for a set $A \in X$ is Lipschitz with global Lipschitz constant 1. Therefore its Clarke directional derivative exists and is finite at every point in all directions. Based on this observation one defines the tangent cone at a point $x \in A$ by

$$T_A(x) := \{ d \in X : a(x; d) \leq 0 \}.$$ 

The polar of the tangent cone is called the normal cone and is denoted by

$$N_A(x) := T_A(x)^0.$$ 

Finally, the contingent cone to $A$ at a point $x \in A$ is given by

$$K_A(x) := \{ d \in X : \exists t_k \downarrow 0, d^k \to d, \text{ with } x + t_k d^k \in A \}.$$ 

2. THE CONVEX CASE

In this section we assume that the set $C$ is nonempty, closed, and convex. In this case the subdifferential properties of $d_C$ can be derived from the observation that $d_C$ is the infimal convolution of the norm and the indicator function for $C$ (see [9]).

**Theorem 1.** Let $C$ be a non-empty, closed, and convex subset of $X$. Then $d_C$ is a convex function on $X$ with convex subdifferential

$$\partial d_C(x) = \begin{cases} \text{bdry}(B) \cap N_C(x) & \text{if } x \notin C, \\ B^0 \cap N_C(x) & \text{if } x \in C, \end{cases}$$

where $C$, is used to denote the set $C + d_C(x)B$. If it is further assumed that $X$ is a reflexive Banach space, then the set of points

$$P_C(x) := \{ x \in C : d_C(x) = \| x - x \| \}$$

is nonempty and for any $\bar{x} \in P_C(x)$ one has the formulas

$$\partial d_C(x) = \partial \| x - \bar{x} \| \cap N_C(x)$$

$$= \partial \| x - \bar{x} \| \cap N_C(\bar{x}).$$

**Proof.** For the case in which $x \in C$ formula (1) is well known. The other half of (1) is established in [3, Sect. 2]. The fact that the set $P_C(x)$ is non-empty when $X$ is reflexive is classical. Formula (3) is an immediate consequence of [5, Lemma 3.33]. From the definition of $\partial \| \cdot \|$ and formula (1), we know that the right hand side of formula (2) is contained in $\partial d_C(x)$. Thus, we need only show that the right hand side of (3) is contained in the
right hand side of (2). Indeed, for every $w \in C$, $z \in B$, and $u \in \partial \|x - \bar{x}\| \cap N_C(\bar{x})$, we have

$$
\langle u, w + d_C(x)z - x \rangle = \langle u, \bar{x} - x \rangle + \langle u, w - \bar{x} \rangle + d_C(x)\langle u, z \rangle
$$

$$
\leq -d_C(x) + d_C(x)\langle u, z \rangle + \langle u, w - \bar{x} \rangle
$$

$$
\leq \langle u, w - \bar{x} \rangle
$$

$$
\leq 0,
$$

whereby the inclusion follows and the result is established. 1

**Corollary 2.** Let $C$ and $X$ be as in Theorem 1. Then $d_C'(x; h)$ exists for all $x$ and $h$ in $X$ and

$$
d_C'(x; h) \leq d_{T_C(x)}(h). \tag{4}
$$

Moreover, if $x \in C$, then equality holds,

$$
d_C'(x; h) = d_{T_C(x)}(h). \tag{5}
$$

**Proof.** We recall from [4, Theorem 3.1] that if $K \subseteq X$ is a nonempty, closed, and convex cone, then

$$
d_K(x) = \psi_{K_0 \cap C_0}(x).
$$

The result now follows from formula (1) and the fact that

$$
d_C'(x; h) = \psi_{C_0 \cap T_C(x)}(h). \tag{7}
$$

3. **The Nonconvex Case: $x \in C$**

In the remainder of the paper we drop the assumption that $C$ is convex and assume only that $C$ is a nonempty closed subset of $X$. Let us first study the case where $x \in C$. Since $d_C$ is Lipschitz with constant 1, we know from Clarke [6, Proposition 2.1.2] that $\partial d_C(x) \subseteq B^0$. Again by Clarke [6, Proposition 2.4.2], we have $\partial d_C(x) \subseteq N_C(x)$. Therefore it is always the case that

$$
\partial d_C(x) \subseteq B^0 \cap N_C(x), \tag{6}
$$

or equivalently,

$$
d_C^0(x; \cdot) \leq d_{T_C(x)}(\cdot). \tag{7}
$$

However, in general, one cannot complete this inclusion to an equation as in formula (1).
Example 3. Let $C \subset \mathbb{R}^2$ be the set $\mathbb{R}_{+}^{2} \cup \mathbb{R}_{-}^{2}$ under the Euclidean norm. Then it is straightforward to show that $B_{0}^{1} \cap N_{C}(x) = \{(u, v) : u^2 + v^2 \leq 1\}$ at $x = (0, 0)$, but $\delta d_{C}(x) = \{(u, v) : |u| + |v| \leq 1\}$.

Although it is not possible to obtain a general formula paralleling formula (1) for the Clarke subdifferential and tangent cone, it is possible, at least in finite dimensions, to obtain a closely related formula for the contingent cone and contingent derivative. Observe that in the convex case, with $x \in C$, formulas (1) and (5) are equivalent. This is not so for non-convex sets $C$ when one uses the contingent cone and the contingent derivative, since neither is assured to be convex. Nonetheless, in the contingent calculus, one can establish (4), and in finite dimensions (5).

Theorem 4. If $x \in C$, then

$$d_{C}^{-}(x; h) \leq d_{K_{c}(x)}(h),$$

for all $h \in X$. If it is further assumed that $X$ is finite dimensional, then equality holds in (8).

Remark. It is an open question as to whether or not (8) can be established as an equality in infinite dimensions.

Proof. Choose $h \in X$ and let $v \in K_{c}(x)$ be such that $\|h - v\| + \varepsilon \leq d_{K_{c}(x)}(h)$. Then there exists $v_{n} \in t_{n}^{-1}(C - x)$ with $t_{n} \downarrow 0$ and $v_{n} \to v$. Hence

$$d_{C}^{-}(x; h) \leq \liminf_{n \to \infty} d_{(t_{n}^{-1}(C - x))(h)}(h)$$

$$\leq \lim_{n \to \infty} \|h - v_{n}\|$$

$$= \|h - v\|$$

$$\leq d_{K_{c}(x)}(h) + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ establishes inequality (8).

Next assume that $X$ is finite dimensional and let $(h_{n}, t_{n})$ be an attaining sequence for the definition of $d_{C}^{-}(x; h)$. Let $v_{n} \in t_{n}^{-1}(C - x)$ be such that

$$d_{(t_{n}^{-1}(C - x))(h_{n})} \geq \|h_{n} - v_{n}\| - t_{n}.$$

Then

$$d_{C}^{-}(x; h) = \liminf_{n \to \infty} \frac{d_{C}(x + t_{n}h_{n}) - d_{C}(x)}{t}$$

$$= \liminf_{n \to \infty} d_{(t_{n}^{-1}(C - x))(h_{n})}$$

$$\geq \liminf_{n \to \infty} \|h_{n} - v_{n}\|. $
Since $d_C(x; h) < \infty$ by formula (8) and $\{h_n\}$ converges, the sequence $\{v_n\}$ is bounded. Thus some subsequence converges to $v \in K_C(x)$ and so

$$d_C(x; h) \geq \|h - v\| \geq d_{K_C(x)}(h).$$

The following corollary is immediate.

**Corollary 5.** If $x \in C$, then

$$K_C(x) \subset \{h : d_C(x; h) \leq 0\},$$

with equality holding if $X$ is finite dimensional.

In the finite dimensional case it is possible to establish a result paralleling formula (1) for the Clarke subdifferential if one invokes a regularity condition.

**Definition 6.** Let $x \in S \subset X$. We say that $S$ is regular at $x$ if

$$T_S(x) = K_S(x).$$

Let $f : X \mapsto \mathbb{R} \cup \{+\infty\}$. We say that $f$ is regular at $x \in X$ if

$$f^o(x; \cdot) = f^-(x; \cdot).$$

**Proposition 7.** Let $C$ be a nonempty closed subset of $X$. If $d_C$ is regular at $x \in C$, then $C$ is regular at $x$.

**Proof.** From inclusion (9) and the definition of $T_C(x)$ we have

$$K_C(x) \subset \{h : d_C(x; h) \leq 0\}$$

$$= \{h : d^o_C(x; h) \leq 0\}$$

$$\subset T_C(x)$$

$$\subset K_C(x).$$

**Theorem 8.** Let $C$ be a nonempty closed subset of $X$. If $X$ is finite dimensional and $d_C$ is regular at $x \in C$, then

$$\partial d_C(x) = B^0 \cap N_C(x)$$

and

$$d^o_C(x; \cdot) = d_{T_C(x)}(\cdot).$$
Proof. By the previous proposition, $K_C(x) = T_C(x)$ and by Theorem 4, $d_C^-(x; \cdot) = d_{K_C(x)}(\cdot)$. This establishes Eq. (11). Equation (10) now follows by convexity. 

**COROLLARY 9.** Let $C$ be a nonempty closed subset of $X$ and let $x \in C$. If $X$ is finite dimensional, then $C$ is regular at $x$ if and only if $d_C$ is regular at $x$.

Proof. The fact that the regularity of $d_C$ implies the regularity of $C$ at $x$ has already been established in Proposition 7. On the other hand if $C$ is regular at $x$, then, by (7) and Theorem 4,

$$d_C^-(x; h) \leq d_C^0(x; h) \leq d_{T_C(x)}(h)$$

$$= d_{K_C(x)}(h) = d_C(x; h),$$

whereby the result follows. 

It is bothersome that we cannot extend Corollary 9 to the infinite dimensional case. The deficiency in our argument resides in the fact that we cannot establish equality in (8) as can be done for finite dimensions. If one could resolve this issue in the affirmative, then the results of this section could be sharpened. However, we are skeptical of the validity of (8) in infinite dimensions.

4. THE NONCONVEX CASE: $x \notin C$

We now concentrate on the case in which $x \notin C$. In this case Borwein, Fitzpatrick, and Giles [1, Theorem 8] establish a very useful characterization of the subdifferential $\partial d_C(x)$ when the norm on $X$ is assumed to be uniformly Gâteaux differentiable (see [1, p. 521]). Their characterization provides a strong approximation property for $\partial d_C(x)$. Although our results are somewhat weaker, they do not require the uniform Gâteaux differentiability hypothesis. Many of these results will be derived from the following elementary fact.

**LEMMA 10.** Let $C$ be a nonempty closed subset of the normed linear space $X$. Then for every $x \in X$ and $\alpha \geq 0$ one has

$$d_C(x) \leq d_{(C + \alpha B)}(x) + \alpha.$$

Proof. Let $u \in B$. Then, by the Lipschitz continuity of $d_C$, we have

$$d_C(x) = d_C((x + \alpha u) - (xu))$$

$$\leq d_C(x + \alpha u) + \alpha.$$
Hence
\[
d_C(x) \leq \left[ \inf_{u \in B} d_C(x + au) \right] + x
= d_{C + xB}(x) + x.
\]

By taking \( \alpha = d_C(x) \) in the above lemma one obtains the relation
\[
d_C(y) \leq d_C(y) + d_C(x). \quad (12)
\]

This inequality allows us to extend several of the results of the previous section.

**Theorem 11.** Let \( C \) be a nonempty closed subset of \( X \). Then for every \( x \in X \), one has
\[
d_C^-(x; \cdot) \leq d_{K_C(x)}(\cdot), \quad (13)
\]
\[
d_C^0(x; \cdot) \leq d_{T_C(x)}(\cdot), \quad (14)
\]
\[
K_C(x) \subseteq \{ h : d_C^-(x; h) \leq 0 \}, \quad (15)
\]

and
\[
\partial d_C(x) = B^0 \cap N_C(x). \quad (16)
\]

**Proof.** When \( x \in C \), this result has already been established in the previous section. Regardless, from (12) we always have the inequalities
\[
d_C^-(x; \cdot) \leq d_C^-(x; \cdot),
\]
and
\[
d_C^0(x; \cdot) \leq d_C^0(x; \cdot).
\]

Inequalities (13) and (14) are now an immediate consequence of these inequalities and those in (8) and (7). The inclusion (15) follows directly from (13) and inclusion (16) follows from (14) and [4, Theorem 3.1] as in Corollary 2. \( \blacksquare \)

**Remark.** One should compare this result with [1, Theorem 8]. Although our inclusion (16) is weaker than this result, we do not restrict the norm on \( X \) to be uniformly Gâteaux differentiable.

The remaining results in this section depend upon the nonemptiness of the set
\[
P_C(x) := \{ \hat{x} \in C : \| x - \hat{x} \| = d_C(x) \}.\]
In the finite dimensional case, $P_C(x)$ is always nonempty as long as $C$ is nonempty and closed. We have the following extension to formula (3) in Theorem 1.

**Theorem 12.** Let $C$ be a nonempty closed subset of $X$ and let $x \in X$ be such that $P_C(x)$ is nonempty. Then

$$\partial d_C(x) \subseteq B^0 \cap \left( \bigcap_{y \in P_C(x)} N_C(y) \right)$$
(17)

$$\subseteq B^0 \cap N_C(\tilde{x})$$
(18)

and

$$d^0_C(x; h) \leq d_{T_C(x)}(h)$$
(19)

for every $\tilde{x} \in P_C(x)$.

**Proof.** We show that $T_{C}(\tilde{x}) \subseteq T_{C}(x)$. Let $v \in T_{C}(\tilde{x})$ and suppose $x_i \to x$ and $t_i \downarrow 0$. Then $x_i + \tilde{x} - x \to \tilde{x}$, so that there exists $v_i \to v$ with $x_i + \tilde{x} - x + t_i v_i \in C$. It follows that

$$x_i + t_i v_i \in C + (x - \tilde{x}) \subseteq C + d_C(x) B$$

so that $v \in T_{C}(x)$. Thus $d_{T_{C}(\tilde{x})}(h) \leq d_{T_{C}(\tilde{x})}(h)$. Formula (19) follows from (14). Formulas (17) and (18) now follow from (19) and [4, Theorem 3.1].

Formula (1) is now extended by assuming that $P_C(x)$ is nonempty and that $d_C$ is regular at $x$. We begin with the following preliminary result.

**Proposition 13.** Let $C$ be a nonempty closed subset of $X$ and let $\tilde{x} \in P_C(x)$. Then

$$(1 - t) d_C(x) = d_C(x + t(\tilde{x} - x))$$
(20)

for all $t \in [0, 1]$. Moreover, $d'_C(x; \tilde{x} - x)$ exists and we have

$$d'_C(x; \tilde{x} - x) = d_C(x; \tilde{x} - x) = -d_C(x).$$
(21)

**Proof.** To see (20) simply observe that

$$d_C(x) = d_C(x + t(\tilde{x} - x) - t(\tilde{x} - x))$$
$$\leq d_C(x + t(\tilde{x} - x)) + td_C(x),$$
and so \((1 - t) d_C(x) \leq d_C(x + t(\bar{x} - x))\). Conversely,

\[
d_C(x + t(\bar{x} - x)) = d_C(\bar{x} + (1 - t)(x - \bar{x})) \leq (1 - t) d_C(x).
\]

The fact that \(d'_C(x; \bar{x} - x)\) exists and equals \(-d_C(x)\) follows immediately from (20). In order to verify the second equality in (21) let \((h_n, t_n) \to (\bar{x} - x, 0)\) be a sequence achieving the limit in the definition of \(d_C'(x; \bar{x} - x)\). Then

\[
d_C'(x; \bar{x} - x) = \lim_{n} \frac{d_C(x + t_n h_n) - d_C(x)}{t_n} \leq \lim_{n} \frac{d_C(x + t_n (\bar{x} - x)) + t_n \| h_n - (\bar{x} - x) \| - d_C(x)}{t_n} = \lim_{n} \frac{(1 - t_n) d_C(x) - d_C(x) + t_n \| h_n - (\bar{x} - x) \|}{t_n} = -d_C(x).
\]

The reverse inequality is obtained in exactly the same way, except now one observes that

\[
d_C'(x; \bar{x} - x) \geq \lim_{n} \frac{d_C(x + t_n (\bar{x} - x)) - t_n \| h_n - (\bar{x} - x) \| - d_C(x)}{t_n}.
\]

**Theorem 14.** Let \(C\) be a nonempty closed subset of \(X\) and let \(x \in X \setminus C\) be such that \(d_C\) is regular at \(x\) and \(P_C(x)\) is nonempty. Then \(C_x := C + d_C(x) \Bbb B\) is regular at \(x\) and for each \(x \in P_C(x)\) we have

\[
d_C(x) = \partial d_C(x) = \partial \| x - \bar{x} \| \cap N_C(x)
\]

(22)

\[
= \partial \| x - \bar{x} \| \cap N_C(x)
\]

(23)

\[
= \left( \bigcap_{\bar{y} \in P_C(x)} \partial \| x - \bar{y} \| \cap N_C(x) \right)
\]

(24)

\[
\subseteq \bigcap_{\bar{y} \in P_C(x)} \left( \partial \| x - \bar{y} \| \cap N_C(\bar{y}) \right)
\]

(25)

\[
\subseteq \partial \| x - \bar{\bar{x}} \| \cap N_C(x).
\]

(26)
Proof. The regularity of $d_c$ at $x$ and Proposition 13 imply that $0 \notin \partial d_c(x)$. Consequently, Clarke [6, Theorem 2.4.7] and (15) imply that

$$K_{C_x}(x) \subseteq \{ h : d_c^-(x; h) \leq 0 \}$$

$$= \{ h : d_c^0(x; h) \leq 0 \}$$

$$\subseteq T_{C_x}(x).$$

Hence $C_x$ is regular at $x$. We now show that $\partial d_c(x) \subseteq \partial \|x - \tilde{x}\|$ for any $\tilde{x} \in P_C(x)$. Indeed, regularity and (21) imply that $d_c^0(x; \tilde{x} - x) = -d_c(x)$, or equivalently,

$$\|x - \tilde{x}\| \leq \langle z, x - \tilde{x} \rangle \quad \forall z \in \partial d_c(x),$$

for every $\tilde{x} \in P_C(x)$. Hence

$$\partial d_c(x) \subseteq \partial \|x - \tilde{x}\| \subseteq \text{bdry}(\mathbb{B}^0).$$

Therefore, by (16),

$$\partial d_c(x) \subseteq \partial \|x - \tilde{x}\| \cap N_{C_x}(x)$$

$$\subseteq \text{bdry}(\mathbb{B}^0) \cap N_{C_x}(x).$$

But, by Clarke [6, Corollary 1, p. 56], we have

$$N_{C_x}(x) \subseteq \bigcup_{\lambda \geq 0} \lambda \partial d_c(x),$$

and so

$$\text{bdry}(\mathbb{B}^0) \cap N_{C_x}(x) = \partial d_c(x),$$

whereby (22) and (23) are established. Formula (24) follows immediately from formula (23). Formulas (25) and (26) follow from (23) and (18).

Remark. It is interesting to compare this result with [1, Theorem 5]. In their result Borwein, Fitzpatrick, and Giles show that the Michel-Penot subdifferential [8] of $d_c$ at $x$ always contains an element of $\partial \|x - \tilde{x}\|$. Under the regularity hypothesis, the Michel-Penot and Clarke subdifferentials coincide and the stronger statement (23) is possible.

It should be observed that the formulas (22)–(24) are only valid when $0 \notin \partial d_c(x)$. In general, this condition does not hold at points $x$ not in $C$ if $d_c$ is not regular at $x$. 
Example 15. Let $X = \mathbb{R}^n$ be given the Euclidean norm and set $C := X \setminus B$. Then $\partial d_C(0) = \mathbb{R}^0$.

In finite dimensions, recall that if $C$ is regular at $x \in C$, then $d_C$ is regular at $x$. A similar result for $C_0$ and $d_C$ will not in general be valid when $x \not\in C$. This is also illustrated by Example 15.

If $X$ has a smooth norm, Theorem 14 yields an interesting characterization of regularity.

**Theorem 16.** Suppose that the norm on $X$ is smooth. Let $C$ be a nonempty closed subset of $X$ and let $x \in X \setminus C$ be such that $P_C(x) \neq \emptyset$. Then the following statements are equivalent:

1. The function $d_C$ is strictly differentiable at $x$.
2. The function $d_C$ is regular at $x$.
3. There exists $\bar{x} \in P_C(x)$ such that

$$d_C^0(x; x - x) = -d_C(x). \quad (27)$$

Furthermore, if any of the above holds, then $P_C(x)$ is a singleton set, and

$$\partial d_C(x) = \{ (x - \bar{x})/\|x - \bar{x}\| \}.$$

**Proof.** The implication $(1) \Rightarrow (2)$ is a consequence of Clarke [6, Proposition 2.2.4, p. 33]. The implication $(2) \Rightarrow (3)$ follows from Proposition 13. The smoothness of the norm and inclusion $(27)$ imply that $P_C(x)$ is a singleton with $\partial d_C(x) = \{ (x - \bar{x})/\|x - \bar{x}\| \}$.

It remains to show that $(3) \Rightarrow (1)$. To this end, let us suppose that Eq. (27) holds for some $\bar{x} \in P_C(x)$. Then

$$\|x - \bar{x}\| \leq \langle z, x - \bar{x} \rangle \quad \forall z \in \partial d_C(x).$$

Hence

$$\partial d_C(x) \subset \partial \|x - \bar{x}\|.$$

Consequently, $\partial d_C(x)$ is a singleton and so, again by Clarke [6, Proposition 2.2.4, p. 33], $d_C$ is strictly differentiable at $x$ with strict derivative $(x - \bar{x})/\|x - \bar{x}\|$. \[\]

The following corollary is an immediate consequence of Theorem 16.

**Corollary 17.** Let $X$, $C$, and $x$ be as in Theorem 16. If $P_C(x)$ contains more than a single element, then $d_C$ is not regular at $x$. 
5. CONCLUDING REMARKS

The goal of this paper was to determine the extent to which the formulas (1), (2), and (3), valid for convex sets $C$, could be extended to nonconvex sets. For the case in which $x \in C$, formula (1) could only be extended when $\mathbb{X}$ is assumed to be finite dimensional and $C$ is regular at $x$. The problem here is our inability to establish inclusion (9) as an equality in infinite dimensions. Although this inclusion holds as an equality in finite dimensions, it is unknown whether this is so in infinite dimensions. We do not believe that it is. If it is not, then regularity does not imply equality in inclusion (6). This would be somewhat surprising, since then the regularity of $C$ at a point $x \in C$ is not equivalent to the regularity of $d_c$ at $x$.

When $x \notin C$ we have seen that the basic results can be derived from the case $x \in C$ (Lemma 10 and Theorem 11). However, the full extension of formulas (1) and (2) to the nonconvex case required the nonemptiness of $P_C(x)$ and the regularity of $d_c$ at $x$ (Theorem 14). The proof technique used to establish these results revealed an interesting characterization of the regularity of $d_c$ at $x$ when the norm on $\mathbb{X}$ is smooth. That is, $d_c$ is regular at $x$ if and only if $d_c$ is strictly differentiable at $x$ whenever $P_C(x)$ is non-empty (which is always the case in finite dimensions).

The efforts to extend formula (3) were not as successful. The reason for this may be due to the inappropriateness of this formula in the nonconvex case. Indeed, in general, the boundary of the set $C$, when $x \notin C$ is better behaved than the boundary of $C$. In order to give the reader some insight into the problem, we give the following example.

Example 18. Let $C \subseteq \mathbb{R}^2$ be given by

$$\left\{ (\xi_1, \xi_2) : \xi_2 = \xi_1 \sin \left( \frac{1}{\xi_1} \right), \xi_1 > 0 \right\} \cup \{(0, 0)\}.$$ 

Then at any point $x = (\xi_1, 0)$ with $\xi_1 < 0$ one has that $d_c$ is regular at $x$ and $P_C(x) = \left\{ \bar{x} \right\}$ where $\bar{x} = (0, 0)$, but $C$ is not regular at $\bar{x}$.

REFERENCES


