# Convex Analysis and Optimization
## SIXTH HOMEWORK SET

Due Monday, July 6, 2015.

(1) Let $H \in \mathbb{R}^{n\times n}_s$, $u \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$ where $\mathbb{R}^{n\times n}_s$ is the linear space of all real symmetric $n \times n$ matrices. Recall that $H$ is said to be *positive definite* if $x^T Hx > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$. Moreover, $H$ is said to be *positive semi–definite* if $x^T Hx \geq 0$ for all $x \in \mathbb{R}^n$. We consider the block matrix

$$\hat{H} := \left[\begin{array}{cc} H & u \\ u^T & \alpha \end{array}\right].$$

  (a) Show that $\hat{H}$ is positive semi–definite if and only if $H$ is positive semi–definite and there exists a vector $z \in \mathbb{R}^n$ such that $u = Hz$ and $\alpha \geq z^T Hz$.

  (b) Show that $\hat{H}$ is positive definite if and only if $H$ is positive definite and $\alpha > u^T H^{-1}u$.

  (c) Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $\delta \in \mathbb{R}$. Use either Part (a) or Part (b) to show that $x \in \mathbb{R}^n$ is a solution to the quadratic inequality

$$(Ax + b)^T (Ax + b) \leq c^T x + \delta$$

  if and only if the block matrix

$$\left[\begin{array}{cc} I & (Ax + b) \\ (Ax + b)^T & (c^T x + \delta) \end{array}\right]$$

  is positive semi–definite.

  (d) Suppose $H$ is positive definite. Show that

$$\left[\begin{array}{cc} H & u \\ 0 & (\alpha - u^T H^{-1}u) \end{array}\right] = \left[\begin{array}{cc} I & 0 \\ (-H^{-1}u)^T & 1 \end{array}\right] \left[\begin{array}{cc} H & u \\ u^T & \alpha \end{array}\right].$$

  (e) Recall that the *kth principal minor* of a matrix $B \in \mathbb{R}^{n\times n}$ is the determinant of the upper left–hand corner k×k–submatrix of $B$ for $1 \leq k \leq n$. Use an induction argument and Parts (b) and (d) above to show that $H$ is positive definite if and only if every principal minor of $H$ is positive.
  Note: Your argument **must** use either Part (a) or Part (b) above.
  Hint: det($AB$)=det($A$)det($B$), and the determinant of an upper or lower block triangular matrix is the product of the determinants of the diagonal blocks.

(2) Show that the mapping $\phi : \mathbb{E} \mapsto \bar{\mathbb{R}}$ is subadditive if and only if $\operatorname{epi}(\phi)$ is a convex cone.

(3) Given a set $S \subset \mathbb{E}$, the *support functional* associated with $S$ is the function $\sigma_S(x) := \sup\{\langle y, x\rangle \mid y \in S\}$.

  (a) Show that $\sigma_S$ is a sublinear convex function.

  (b) Set $D := \operatorname{cl}(\operatorname{conv}(S))$. Show that $\sigma_S = \sigma_D$.

(4) Let $C \subset \mathbb{E}$ be convex. We define the *relative interior* of $C$, denoted $\operatorname{ri}(C)$, to be its interior relative to its affine hull, i.e.,

$$\operatorname{ri}(C) : \{x \in C \mid \exists\, \epsilon > 0 \text{ s.t. } (x + \epsilon\mathbb{B}) \cap \operatorname{aff}(C) \subset C\}.$$

Here, recall that $\operatorname{aff}(C)$ is the smallest affine set containing $C$. Show that $x \in \operatorname{ri}(C)$ if and only if for all $z \in C$ there exists a $\bar{\lambda} > 1$ such that $z + \bar{\lambda}(x - z) \in C$.

(5) Let $f : \mathbb{E} \mapsto \bar{\mathbb{R}}$ be a proper convex function. Show that

$$\operatorname{ri}(\operatorname{epi}(f)) = \{(x, \mu) \mid x \in \operatorname{ri}(\operatorname{dom}(f)) \quad \text{and} \quad f(x) < \mu\}.$$

(6) Let $C \subset \mathbb{E}$ be a nonempty closed convex set and let $P_C$ denote the projection onto $C$ in the inner product norm on $\mathbb{E}$. Show that

  (a) $\|P_C x - P_C y\|_2^2 \leq \langle P_C x - P_C y, x - y\rangle \quad \forall\, x, y \in \mathbb{E}$, and

  (b) $\|P_C x - P_C y\|_2 \leq \|x - y\|_2 \quad \forall\, x, y \in \mathbb{E}$.

  (c) $\|P_C x - P_C y\|_2^2 + \|(I - P_C)x - (I - P_C)y\|_2^2 \leq \|x - y\|_2^2 \quad \forall\, x, y \in \mathbb{E}$.

(7) Let $C \subset \mathbb{E}$ be non-empty closed and convex and consider the support function for $C$:

$$\delta^*_C(x) = \sigma_C(x) := \sup_{y\in C} \langle y, x\rangle \ .$$

  (a) Given $x \in \mathbb{E}$, show that $\operatorname*{argmax}_{y\in C} \langle y, x\rangle$ is a closed convex set whenever it is nonempty.

  (b) If $\operatorname*{argmax}_{y\in C} \langle y, x\rangle$ is nonempty, show that $x$ separates every element of $\operatorname*{argmax}_{y\in C} \langle y, x\rangle$ from $C$.

  (c) If $\operatorname*{argmax}_{y\in C} \langle y, x\rangle$ is nonempty, show that $x \in N(z \,|C)$ for all $z \in \operatorname*{argmax}_{y\in C} \langle y, x\rangle$.

  (d) Show that if $\operatorname{ri}(C) \cap \operatorname*{argmax}_{y\in C} \langle y, x\rangle \neq \emptyset$, then $C = \operatorname*{argmax}_{y\in C} \langle y, x\rangle$.

  (e) Show that the subdifferential of the support function for $C$ at $x$, $\partial\delta^*_C(x)$, is the set $\operatorname*{argmax}_{y\in C} \langle y, x\rangle$.

(8) Let $Q \in \mathbb{S}^n_+$, $c\mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. Let $\|\cdot\|$ denote a norm on $\mathbb{R}^k$, and let $\|\cdot\|_p$ denote the $p$-norm on $\mathbb{R}^k$.

- (a) Show that if $c \in \mathrm{Ran}(Q)$, then, for $k = \mathrm{rank}(Q)$, there is a matrix $M \in \mathbb{R}^{k\times n}$, a vector $d \in \mathbb{R}^k$, and an $\eta \in \mathbb{R}$ such that
$$\frac{1}{2}x^TQx + c^Tx + \alpha = \frac{1}{2}\|Mx+d\|_2^2 + \eta.$$
- (b) Compute the Fenchel-Rockafellar dual for each of the following optimization problems.
  - (i) $\min\left\{\frac{1}{2}x^TQx + c^Tx + \alpha \mid Ax \le b\right\}$, when $Q \in \mathbb{S}^n_{++}$.
  - (ii) $\min\left\{\frac{1}{2}x^TQx + c^Tx + \alpha \mid \|Ax-b\| \le \delta\right\}$, when $Q \in \mathbb{S}^n_{++}$.
  - (iii) $\min\left\{\|Ax-b\| \mid \frac{1}{2}x^TQx + c^Tx \le \beta\right\}$, when $c \in \mathrm{Ran}(Q)$. (Hint: use (a))

(9) Consider the function $\mathrm{maxsum}_k : \mathbb{R}^n \mapsto \mathbb{R}$ given by
$$\mathrm{maxsum}_k(x) := \max\left\{\sum_{i\in I}^{k} x_i \mid I \text{ contains } k \text{ distinct integers from } \{1,2,\dots,n\}\right\}.$$
Note that the function $\mathrm{vecmax} : \mathbb{R}^n \mapsto \mathbb{R}$ given by $\mathrm{vecmax}(x) := \max\{x_i \mid i = 1,2,\dots,n\}$ coincides with $\mathrm{maxsum}_1$. Show that for $k = 1,2,\dots,n$,
$$\mathrm{maxsum}_k(x) = \delta^*(x \mid \Delta_k^{n-1}), \quad \text{where} \quad \Delta_k^{n-1} := \left\{y \in \mathbb{R}^n_+ \;\middle|\; y \le \mathrm{e} \text{ and } \mathrm{e}^Ty = k\right\}$$
with e being the vector or all ones of the required dimension, e.g. in $\mathbb{R}^3$ we have $\mathrm{e} = (1,1,1)^T$.

(10) Let $f : \mathbb{E} \mapsto \bar{\mathbb{R}}$ be closed proper convex. Show that the mapping $\partial f$ is closed, i.e.
$$z^k \in \partial f(x^k),\ x^k \to x,\ z^k \to z \implies z \in \partial f(x).$$

(11) Let $C, D \subset \mathbb{E}$ be nonempty closed convex sets.

- (a) Show that $\mathrm{dom}\,(\delta^*_D)$ is a convex cone.
- (b) Show that $\bigcup_{x\in D} N\left(x\,|D\right) = \mathrm{dom}\,(\partial\delta^*_D) \subset \mathrm{dom}\,(\delta)^*_C \subset \mathrm{cl}\left(\bigcup_{x\in D} N\left(x\,|D\right)\right)$.
- (c) Show that $(\mathrm{dom}\,(\delta^*_D))^\circ = D^\infty = \left(\bigcup_{x\in D} N\left(x\,|D\right)\right)^\circ$.
- (d) Show that $\mathrm{dom}\,(\gamma_C) = \mathrm{cone}\,(C)$, where $\gamma_C$ is the gauge function of $C$ and $\mathrm{cone}\,(C) = \{\lambda x \mid 0 \le \lambda \text{ and } x \in C\}$.
- (e) Recall from the fourth problem in the homework set that $\gamma_C(x) = \delta^*_{C^\circ}(x)$ whenever $0 \in C$. Use this and the results above to show that if $0 \in C$, then $(\mathrm{cone}\,(C^\circ))^\circ = C^\infty$. (There are other ways to show this, indeed, it can be shown directly. You may do this if you choose.)

(12) Let $K_1 \in \mathbb{E}_1$ and $K_2 \in \mathbb{E}_2$ be nonempty closed convex cones, and let $A \in L(\mathbb{E}_1, \mathbb{E}_2)$.

- (a) When $\mathbb{E}_1 = \mathbb{E}_2$, show that $(K_1 + K_2)^\circ = K_1^\circ \cap K_2^\circ$.
- (b) Show that $(AK_1)^\circ = (A^T)^{-1}K_1^\circ$.

(13) Let $C \in \mathbb{E}_1$ and $D \in \mathbb{E}_2$ be nonempty closed convex sets, and let $A \in L(\mathbb{E}_1, \mathbb{E}_2)$. We associate with $A$ the Lagrangian function
$$L(x,y) := \langle y, Ax\rangle + \delta_C(x) - \delta_D(y).$$
Note that $L$ is convex in $x$ and concave in $y$.

- (a) What are the primal and dual objectives and problems associated with $L$?
- (b) [Von Neumans's Minimax Theorem (1948)] Show that if $0 \in \mathrm{intr}\,(\mathrm{dom}\,(\delta^*_D) - AC)$, then
$$\inf_{x\in C}\sup_{y\in D}\langle y, Ax\rangle \;=\; \max_{y\in D}\inf_{x\in C}\langle y, Ax\rangle\,.$$
  This results is the foundational result for matrix games, and the origins of game theory.
- (c) Use the results of problems (11) and (12) to show that
$$0 \in \mathrm{intr}\,(\mathrm{dom}\,(\delta^*_D) - AC) \iff \{0\} = D^\infty \cap (A^T)^{-1}(\mathrm{cone}\,(-C))^\circ\,.$$
- (d) Use symmetry to show that if $0 \in \mathrm{intr}\left(\mathrm{dom}\,(\delta^*_C) + A^TD\right)$, then
$$\min_{x\in C}\sup_{y\in D}\langle y, Ax\rangle \;=\; \sup_{y\in D}\inf_{x\in C}\langle y, Ax\rangle\,.$$
  (Hint: Multiply the minimax problem in part (b) by $-1$.)
- (e) Show that if either $C$ or $D$ is compact, then there is zero duality gap, and if both are compact then the *sup*s become *max* and the *inf*s become *min.*