Convex Analysis and Optimization
FIFTH HOMEWORK SET

(1) Let f and g be elements of $\Gamma := \big\{h : \mathbb{E} \mapsto \bar{\mathbb{R}} \mid h \text{ is closed proper and convex}\big\}$, and write $f \underset{*}{\longleftrightarrow} g$ to denote that f and g are conjugate pairs. Establish the following conjugate relations ships.
   (i) $f(x) - \langle a, x\rangle \underset{*}{\longleftrightarrow} g(v+a)$
   (ii) $f(x+b) \underset{*}{\longleftrightarrow} g(v) - \langle v, b\rangle$
   (iii) $f(x) + c \underset{*}{\longleftrightarrow} g(v) - c$
   (iv) $\lambda f(x) \underset{*}{\longleftrightarrow} \lambda g(v/\lambda) \quad (\lambda > 0)$
   (v) $\lambda f(v/\lambda) \underset{*}{\longleftrightarrow} \lambda g(x) \quad (\lambda > 0)$

(2) Let $Q \in \mathbb{S}^n_+$ (where $\mathbb{S}^n_+$ is the cone of $n \times n$ symmetric positive semi-definite matrices), $a \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, and set $L := \mathrm{Ran}(Q)$. Define $f(x) := \frac{1}{2}x^TQx + \langle a, x\rangle + \alpha$. Suppose Q has eigenvalue decomposition $Q = UDU^T$, where U is orthogonal and $D = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_k, 0, 0, \ldots, 0)$ with $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_k > 0$. The *Moore-Penrose pseudo inverse* of Q is the matrix $Q^\dagger := UD^\dagger U^T$, where $D^\dagger = \mathrm{diag}(\delta_1^{-1}, \delta_2^{-1}, \ldots, \delta_k^{-1}, 0, 0, \ldots, 0)$.
   (i) Show that $QQ^\dagger = Q^\dagger Q$ is the orthogonal projector onto the L.
   (ii) Show that

$$f^*(v) = \begin{cases} \frac{1}{2}(v-a)^TQ^\dagger(v-a) - \alpha & \text{when } \ v \in L + a, \\ +\infty & \text{otherwise.} \end{cases}$$

(3) Let $Q \in \mathbb{S}^n_+,\ c\mathbb{R}^n,\ A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. Let $\|\cdot\|$ denote a norm on $\mathbb{R}^k$, and let $\|\cdot\|_p$ denote the p-norm on $\mathbb{R}^k$. Compute the Fenchel-Rockafellar dual for each of the following optimization problems.
   (i) $\min\big\{\frac{1}{2}x^TQx + c^Tx \mid Ax \leq b\big\}$
   (ii) $\min\big\{\frac{1}{2}x^TQx + c^Tx \mid \|Ax - b\| \leq \delta\big\}$
   (iii) $\min\big\{\|Ax - b\| \;\big|\; \frac{1}{2}x^TQx + c^Tx \leq \beta\big\}$

(4) Let $Q \in \mathbb{S}^n_+,\ c\mathbb{R}^n,\ A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. Let $\|\cdot\|$ denote a norm on $\mathbb{R}^k$, and let $\|\cdot\|_p$ denote the p-norm on $\mathbb{R}^k$. Compute the Lagrangian dual for each of the following optimization problems.
   (i) $\min\big\{\frac{1}{2}x^TQx + c^Tx \mid Ax \leq b\big\}$
   (ii) $\min\big\{\frac{1}{2}x^TQx + c^Tx \mid \|Ax - b\|_\infty \leq \delta\big\}$
   (iii) $\min\big\{\|Ax - b\|_1 \;\big|\; \frac{1}{2}x^TQx + c^Tx \leq \beta\big\}$

Note that in problems 3) and 4) above, you must also discuss when these problems take infinite values.