

Convex Analysis and Optimization

FOURTH HOMEWORK SET

- (1) Let $C \subset \mathbb{E}$ be non-empty closed and convex. Compute the subdifferential of the indicator for C ,

$$\delta_C(x) := \begin{cases} 0 & x \in C \\ +\infty & x \notin C. \end{cases}$$

- (2) Let $C \subset \mathbb{E}$ be non-empty closed and convex and consider the support function for C :

$$\delta_C^*(x) := \sup_{y \in C} \langle y, x \rangle .$$

- (a) Given $x \in \mathbb{E}$, show that $\operatorname{argmax}_{y \in C} \langle y, x \rangle$ is a closed convex set whenever it is nonempty.
- (b) If $\operatorname{argmax}_{y \in C} \langle y, x \rangle$ is nonempty, show that x separates every element of $\operatorname{argmax}_{y \in C} \langle y, x \rangle$ from C .
- (c) If $\operatorname{argmax}_{y \in C} \langle y, x \rangle$ is nonempty, show that $x \in N(z|C)$ for all $z \in \operatorname{argmax}_{y \in C} \langle y, x \rangle$.
- (d) Show that if $\operatorname{ri}(C) \cap \operatorname{argmax}_{y \in C} \langle y, x \rangle \neq \emptyset$, then $C = \operatorname{argmax}_{y \in C} \langle y, x \rangle$.
- (e) Show that the subdifferential of the support function for C at x , $\partial \delta_C^*(x)$, is the set $\operatorname{argmax}_{y \in C} \langle y, x \rangle$.

- (3) The *polar* of a set $S \subset \mathbb{E}$ is the set defined by

$$S^\circ : \{z \mid \langle z, y \rangle \leq 1 \ \forall y \in S\} .$$

- (a) If $K \subset \mathbb{E}$ is a nonempty cone, show that

$$K^\circ : \{z \mid \langle z, y \rangle \leq 0 \ \forall y \in K\} ,$$

and that K° is a closed convex cone.

- (b) If $S \subset \mathbb{E}$ is a nonempty subspace, show that $S^\circ = S^\perp$.
- (c) If $K \subset \mathbb{E}$ is a nonempty closed convex cone, show that $\delta_K^* = \delta_{K^\circ}$.
- (d) If $K \subset \mathbb{E}$ is a nonempty cone, show that $(K^\circ)^\circ$ is the smallest closed convex cone containing K .
- (e) If $C \subset \mathbb{E}$ is closed and convex, show that $(C^\circ)^\circ = \operatorname{conv}(C \cup \{0\})$.
- (f) If $\|y\|$ is a norm on \mathbb{E} (not necessarily the Euclidean norm), show that $\|y\| = \delta_{\mathbb{B}^\circ}^*(y)$, where $\mathbb{B} := \{x \mid \|x\| \leq 1\}$ is the closed unit ball of the norm $\|\cdot\|$.
- (g) Let \mathbb{B} be the closed unit ball of a norm $\|\cdot\|$ on \mathbb{E} (not necessarily the Euclidean norm). Show that the closed unit ball of the norm

$$\|z\|_* := \sup_{x \in \mathbb{B}} \langle x, z \rangle$$

(dual to the norm $\|\cdot\|$) is \mathbb{B}° .

- (4) Given a nonempty closed convex set $U \subset \mathbb{E}$ that contains the origin, we define the gauge function for C by

$$\gamma_U(x) := \inf \{t \mid 0 \leq t, x \in tU\} .$$

- (a) If $B \subset \mathbb{E}$ is the closed unit ball associated with some norm on \mathbb{E} , show that $\|x\| = \gamma_B(x)$.
- (b) The polar to the gauge γ_U is the function

$$\gamma_U^\circ(z) := \sup_{\gamma_U(x) \leq 1} \langle z, x \rangle .$$

Show that γ_U and γ_U° satisfy

$$\langle z, x \rangle \leq \gamma_U^\circ(z) \gamma_U(x) \quad \forall x, z \in \mathbb{E} .$$

- (c) Show that γ_U° is the gauge of U° , i.e., $\gamma_U^\circ = \gamma_{U^\circ}$.
- (d) Given a nonempty closed convex set $U \subset \mathbb{E}$ that contains the origin, show that $\gamma_U(x) = \delta_{U^\circ}^*(x)$.