Convex Analaysis and Optimization
THIRD HOMEWORK SET

(1) Let $f : \mathbb{E} \mapsto \bar{\mathbb{R}}$ be a proper convex function. Show that

$$\operatorname{ri}(\operatorname{epi}(f)) = \{(x,\mu) \mid x \in \operatorname{ri}(\operatorname{dom}(f)) \quad \text{and} \quad f(x) < \mu\}.$$

(2) Let $C \subset \mathbb{E}$ be a nonempty closed convex set and let P_C denote the projection onto C in the inner product norm on $\mathbb{E}$. Show that
  (a) $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle \quad \forall\, x, y \in \mathbb{E}$, and
  (b) $\|P_C x - P_C y\| \le \|x - y\| \quad \forall\, x, y \in \mathbb{E}$.

(3) Let $C \subset \mathbb{E}$. Show that the distance function

$$\operatorname{dist}(x \,|C) := \inf\{\|x - y\| \mid y \in C\}$$

is convex if and only if C is a convex subset of $\mathbb{E}$.

(4) Given a set $S \subset \mathbb{E}$, we define the *horizon* of S to be the set

$$S^\infty := \left\{ y \in \mathbb{E} \,\middle|\, \exists\, \{y^k\} \subset S,\ t_k \downarrow 0 \text{ st } t_k y^k \to y \right\}.$$

Show that
  (a) S^∞ is a closed cone.
  (b) S is bounded if and only if $S^\infty = \{0\}$.

(5) Let $C \subset \mathbb{E}$ be a nonempty closed convex set and let $x \in C$. Recall that the tangent and normal cones to C at x are respectively given by

$$\begin{aligned} T(x\,|C) &:= \operatorname{cl}\left(\bigcup_{\lambda > 0} \lambda^{-1}(C - x) \right) \quad \text{and} \\ N(x\,|C) &:= \{z \in \mathbb{E} \mid \langle z, y - x\rangle \le 0 \ \forall\, y \in C\}. \end{aligned}$$

Consider the polyhedral convex set

$$C := \{x \mid \langle a_i, x\rangle \le \alpha_i,\ i = 1, 2, \ldots, s \quad \text{and} \quad \langle a_i, x\rangle = \alpha_i,\ i = s+1, \ldots, m\},$$

where $(a_i, \alpha_i) \in \mathbb{E} \times \mathbb{R},\ i = 1, \ldots, m$. Given $\bar{x} \in C$, describe the sets C^∞, $T(\bar{x}\,|C)$, and $N(\bar{x}\,|C)$.

(6) Let $f : \mathbb{E} \mapsto \bar{\mathbb{R}}$ be closed proper and convex. We have defined the subdifferential and horizon subdifferential for f at a point $x \in \operatorname{dom}(f)$ to be

$$\begin{aligned} \partial f(x) &:= \{w \in \mathbb{E} \mid (w, -1) \in N((x, f(x))\,|\operatorname{epi}(f))\} \quad \text{and} \\ \partial^\infty f(x) &:= \{w \in \mathbb{E} \mid (w, 0) \in N((x, f(x))\,|\operatorname{epi}(f))\}. \end{aligned}$$

  (a) Show that $\partial^\infty f(x) = (\partial f(x))^\infty$ when $\partial f(x) \ne \emptyset$, where $(\partial f(x))^\infty$ is the horizon set for $\partial f(x)$.
  (b) Show that $N(x\,|\operatorname{dom}(f)) \subset \partial^\infty f(x)$.
  (c) Compute both $\partial f(x)$ and $\partial^\infty f(x)$ at $x = 0$ for the functions $f(x) = |x|$ and $f(x) = \sqrt{1 - (x-1)^2}$.

(7) Let $C \subset \mathbb{E}$ be nonempty closed and convex. Given $x \in \operatorname{cl}(C) \setminus \operatorname{ri}(C)$ show that there exists a sequence $\{x^k\} \in \mathbb{E} \setminus \operatorname{cl}(C)$ with $x^k \to x$.