

## Convex Analysis and Optimization

### SECOND HOMEWORK SET

- (1) Show that the function  $\text{logexp} : \mathbb{R}^n \mapsto \bar{\mathbb{R}}$ , given by

$$\text{logexp}(x) := \log(e^{x_1} + e^{x_2} + \cdots + e^{x_n}),$$

is convex.

- (2) Let  $g : \mathbb{E} \mapsto \bar{\mathbb{R}}$  is convex, and let  $\theta : \mathbb{R}_e \mapsto \mathbb{R}_e$  be a nondecreasing function on  $\mathbb{R}$  with  $\theta(-\infty) := \inf \theta$  and  $\theta(+\infty) = +\infty$ . Show that  $\theta \circ g$  is convex.
- (3) Let  $C \subset \mathbb{E}$  be convex, and let  $\lambda_1, \lambda_2 \in \mathbb{R}_+$ . Show that  $(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C$ .
- (4) Let  $K \subset \mathbb{E}$  be closed. Show that  $K$  is a closed convex cone if and only if  $K + K \subset K$  and  $\lambda K \subset K$  for all  $\lambda \geq 0$ .
- (5) Show that the mapping  $\phi : \mathbb{E} \mapsto \bar{\mathbb{R}}$  is subadditive if and only if  $\text{epi}(\phi)$  is a convex cone.
- (6) Given a set  $S \subset \mathbb{E}$ , the *support functional* associated with  $S$  is the function  $\sigma_S(x) := \sup \{\langle y, x \rangle \mid y \in S\}$ .
- (a) Show that  $\sigma_S$  is a sublinear convex function.
- (b) Set  $D := \text{cl}(\text{conv}(S))$ . Show that  $\sigma_S = \sigma_D$ .
- (7) Let  $f : \mathbb{E} \mapsto \bar{\mathbb{R}}$ . Show that  $f$  is lower semi-continuous (lsc) if and only if  $\text{epi}(f)$  is a closed set.
- (8) Let  $C \subset \mathbb{E}$  be convex. We define the *relative interior* of  $C$ , denoted  $\text{ri}(C)$ , to be its interior relative to its affine hull, i.e.,

$$\text{ri}(C) : \{x \in C \mid \exists \epsilon > 0 \text{ s.t. } (x + \epsilon\mathbb{B}) \cap \text{aff}(C) \subset C\}.$$

Here, recall that  $\text{aff}(C)$  is the smallest affine set containing  $C$ . Show that  $x \in \text{ri}(C)$  if and only if for all  $z \in C$  there exists a  $\bar{\lambda} > 1$  such that  $z + \bar{\lambda}(x - z) \in C$ .