Linear Programming

Introduction to Linear Programming

April 23, 2015

What is linear programming (LP)?

What is linear programming (LP)?

A *linear program* is an optimization problem in finitely many variables having a linear objective function and a constraint region determined by a finite number of linear equality and/or inequality constraints.

Compact Representation

$$
\begin{array}{lll}
\text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n & \\
\text{subject to} & a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq \alpha_i & i = 1, \ldots, s \\
& b_{i1}x_i + b_{i2}x_2 + \cdots + b_{in}x_n = \beta_i & i = 1, \ldots, r.
\end{array}
$$

Vector Inequalities: Componentwise

Let $x, y \in \mathbb{R}^n$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

We write $x \leq y$ if and only if

$$x_i \leq y_i, \ \ i = 1, 2, \ldots, n \ .$$

Matrix Notation

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = c^T x$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix Notation

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq \alpha_i \quad i = 1, \ldots, s$$

$$\Longleftrightarrow$$

$$Ax \leq a$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \ldots & a_{sn} \end{bmatrix} \qquad a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_s \end{bmatrix}$$

Matrix Notation

$$b_{i1}x_i + b_{i2}x_2 + \cdots + b_{in}x_n = \beta_i \quad i = 1, \ldots, r$$

$$\iff$$

$$Bx = b$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \ldots & b_{rn} \end{bmatrix} \qquad b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_r \end{bmatrix}$$

LP's Matrix Notation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq a \text{ and } Bx = b \end{array}$$

LP's Matrix Notation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le a \text{ and } Bx = b \end{array}$$

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \; a = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{bmatrix}, \; b = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix}.$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \\ a_{s1} & \dots & a_{sn} \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ & \ddots & \\ b_{r1} & \dots & b_{rn} \end{bmatrix}$$

Applications of Linear Programing

A short list:

Applications of Linear Programing

A short list:

- resource allocation

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling
- facility location

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling
- facility location
- supply chain management

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling
- facility location
- supply chain management
- flight crew scheduling

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling
- facility location
- supply chain management
- flight crew scheduling
- portfolio optimization

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling
- facility location
- supply chain management
- flight crew scheduling
- portfolio optimization
- cash flow matching

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling
- facility location
- supply chain management
- flight crew scheduling
- portfolio optimization
- cash flow matching
- currency exchange arbitrage

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling
- facility location
- supply chain management
- flight crew scheduling
- portfolio optimization
- cash flow matching
- currency exchange arbitrage
- crop scheduling

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling
- facility location
- supply chain management
- flight crew scheduling
- portfolio optimization
- cash flow matching
- currency exchange arbitrage
- crop scheduling
- diet balancing

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling
- facility location
- supply chain management
- flight crew scheduling
- portfolio optimization
- cash flow matching
- currency exchange arbitrage
- crop scheduling
- diet balancing
- parameter estimation

Applications of Linear Programing

A short list:

- resource allocation
- production scheduling
- warehousing
- layout design
- transportation scheduling
- facility location
- supply chain management
- flight crew scheduling
- portfolio optimization
- cash flow matching
- currency exchange arbitrage
- crop scheduling
- diet balancing
- parameter estimation
- ...

Example: Plastic Cup Factory

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Example: Plastic Cup Factory

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Model this problem as an LP.

LP Modeling

The four basic steps of LP modeling.

LP Modeling

The four basic steps of LP modeling.

1. Identify and label the *decision variables*.

LP Modeling

The four basic steps of LP modeling.

1. Identify and label the *decision variables*.
2. Determine the objective and use the decision variables to write an expression for the *objective function*.

LP Modeling

The four basic steps of LP modeling.

1. Identify and label the *decision variables*.
2. Determine the objective and use the decision variables to write an expression for the *objective function*.
3. Determine the *explicit constraints* and write a functional expression for each of them.

LP Modeling

The four basic steps of LP modeling.

1. Identify and label the *decision variables*.
2. Determine the objective and use the decision variables to write an expression for the *objective function*.
3. Determine the *explicit constraints* and write a functional expression for each of them.
4. Determine the *implicit constraints*.

Decision Variables

Determining the decision variables is the most difficult part of modeling.

Decision Variables

Determining the decision variables is the most difficult part of modeling.

To determining these variables it is helpful to put yourself in the shoes of the decision maker, then ask *What must he or she know to do their job*.

Decision Variables

Determining the decision variables is the most difficult part of modeling.

To determining these variables it is helpful to put yourself in the shoes of the decision maker, then ask *What must he or she know to do their job*.

In the *real world*, the modeler often follows the decision maker around for days, weeks, even months at a time recording all of the actions and decisions this person must make.

Decision Variables

Determining the decision variables is the most difficult part of modeling.

To determining these variables it is helpful to put yourself in the shoes of the decision maker, then ask *What must he or she know to do their job*.

In the *real world*, the modeler often follows the decision maker around for days, weeks, even months at a time recording all of the actions and decisions this person must make.

In this phase of modeling, it is very important to resist the temptation to make assumptions about the nature of the solution.

Decision Variables

Determining the decision variables is the most difficult part of modeling.

To determining these variables it is helpful to put yourself in the shoes of the decision maker, then ask *What must he or she know to do their job*.

In the *real world*, the modeler often follows the decision maker around for days, weeks, even months at a time recording all of the actions and decisions this person must make.

In this phase of modeling, it is very important to resist the temptation to make assumptions about the nature of the solution. This last point cannot be over emphasized. Even the most experienced modelers occasionally fall into this trap since such assumptions can enter in very subtle ways.

Decision Variables

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Decision Variables

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is \$25 while the profit on a case of champagne glasses is \$20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

$B =$ number of cases of beer mugs produced daily

Decision Variables

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

$B =$ number of cases of beer mugs produced daily
$C =$ number of cases of champagne glasses produced daily

Objective Function

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is \$25 while the profit on a case of champagne glasses is \$20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Objective Function

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Maximize Profit:

Objective Function

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is \$25 while the profit on a case of champagne glasses is \$20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Maximize Profit: Profit $=$ Revenue $-$ Costs

Objective Function

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Maximize Profit: Profit = Revenue − Costs

$\text{Profit} = 25B + 20C$

Explicit Constraints

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Explicit Constraints

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is \$25 while the profit on a case of champagne glasses is \$20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Resin:

Explicit Constraints

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Resin: $20B \ + \ 12C \ \leq \ 1800$

Explicit Constraints

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is \$25 while the profit on a case of champagne glasses is \$20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Resin: $20B + 12C \leq 1800$

Labor:

Explicit Constraints

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Resin: $20B + 12C \leq 1800$

Labor: $B/15 + C/15 \leq 8$

Implicit Constraints

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Implicit Constraints

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

The decision variables are non-negative:

Implicit Constraints

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

The decision variables are non-negative: $0 \leq B, \; 0 \leq C$

The Plastic Cup Factory LP Model

$$
\begin{aligned}
\text{maximize} \quad & 25B + 20C \\
\text{subject to} \quad & 20B + 12C \leq 1800 \\
& \tfrac{1}{15}B + \tfrac{1}{15}C \leq 8 \\
& 0 \leq B, C
\end{aligned}
$$

The Hardest Part of Modeling: Decision Variables

Once again, the first step in the modeling process, identification of the decision variables, is always the most difficult.

The Hardest Part of Modeling: Decision Variables

Once again, the first step in the modeling process, identification of the decision variables, is always the most difficult.

Never be afraid to add more decision variables either to clarify the model or to improve its flexibility. Modern LP software easily solves problems with tens of thousands of variables, and in some cases tens of millions of variables. It is more important to get a correct, easily interpretable, and flexible model then to provide a compact minimalist model.

The Hardest Part of Modeling: Decision Variables

Once again, the first step in the modeling process, identification of the decision variables, is always the most difficult.

Never be afraid to add more decision variables either to clarify the model or to improve its flexibility. Modern LP software easily solves problems with tens of thousands of variables, and in some cases tens of millions of variables. It is more important to get a correct, easily interpretable, and flexible model then to provide a compact minimalist model.

LP model solutions found in many texts fall into the trap of trying to provide the most compact minimalist model with the fewest possible number of variables and constraints.

The Hardest Part of Modeling: Decision Variables

Once again, the first step in the modeling process, identification of the decision variables, is always the most difficult.

Never be afraid to add more decision variables either to clarify the model or to improve its flexibility. Modern LP software easily solves problems with tens of thousands of variables, and in some cases tens of millions of variables. It is more important to get a correct, easily interpretable, and flexible model then to provide a compact minimalist model.

LP model solutions found in many texts fall into the trap of trying to provide the most compact minimalist model with the fewest possible number of variables and constraints.
Do not repeat this error in developing your own models.

Model 4: Blending

A company makes a blend consisting of two chemicals, 1 and 2, in the ratio 5:2 by weight. These chemical can be manufactured by three different processes using two different raw materials and a fuel. Production data are given in the table below. For how much time should each process be run in order to maximize the total amount of *blend* manufactured?

Model 4: Blending

A company makes a blend consisting of two chemicals, 1 and 2, in the ratio 5:2 by weight. These chemical can be manufactured by three different processes using two different raw materials and a fuel. Production data are given in the table below. For how much time should each process be run in order to maximize the total amount of *blend* manufactured?

	Requirements per Unit Time			*Output per Unit Time*	
Process	*Raw Mat. 1 (units)*	*Raw Mat. 2 (units)*	*Fuel (units)*	*Chem. 1 (units)*	*Chem. 2 (units)*
1	9	5	50	9	6
2	6	8	75	7	10
3	4	11	100	10	6
Amount available	200	400	1850		

Model 4: Blending

What are the decision variables?

Model 4: Blending

What are the decision variables?

$$T_i \; = \; \text{number of time units running process } i = 1, 2, 3.$$

Model 4: Blending

What are the decision variables?

$$T_i \ = \ \text{number of time units running process } i = 1, 2, 3.$$

What is the objective functions?

Model 4: Blending

What are the decision variables?

$$T_i \; = \; \text{number of time units running process } i = 1, 2, 3.$$

What is the objective functions?

$$\text{maximize Blend} \; = \; B$$

Model 4: Blending

Explicit Constraints:

Resources

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Chemicals Produced:

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Chemicals Produced:

Chemical 1 $= C_1 = 9T_1 + 7T_2 + 10T_3$

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Chemicals Produced:

Chemical 1 $= C_1 = 9T_1 + 7T_2 + 10T_3$

Chemical 2 $= C_2 = 6T_1 + 10T_2 + 6T_3$

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Chemicals Produced:

$$\text{Chemical 1} = C_1 = 9T_1 + 7T_2 + 10T_3$$

$$\text{Chemical 2} = C_2 = 6T_1 + 10T_2 + 6T_3$$

Blend:

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Chemicals Produced:

Chemical 1 $=$ $C_1 = 9T_1 + 7T_2 + 10T_3$

Chemical 2 $=$ $C_2 = 6T_1 + 10T_2 + 6T_3$

Blend: $\frac{5}{7}B \leq C_1$

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Chemicals Produced:

Chemical 1 = $C_1 = 9T_1 + 7T_2 + 10T_3$

Chemical 2 = $C_2 = 6T_1 + 10T_2 + 6T_3$

Blend: $\frac{5}{7}B \leq C_1$ and $\frac{2}{7}B \leq C_2$

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Chemicals Produced:

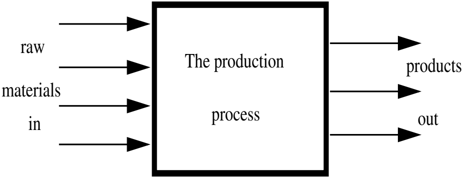
$$\text{Chemical 1} = C_1 = 9T_1 + 7T_2 + 10T_3$$

$$\text{Chemical 2} = C_2 = 6T_1 + 10T_2 + 6T_3$$

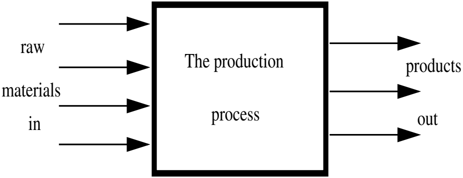
Blend: $\frac{5}{7}B \leq C_1$ and $\frac{2}{7}B \leq C_2$

Implicit Constraints: $0 \leq B,\ 0 \leq T_i,\ i = 1, 2, 3,\ 0 \leq C_j,\ j = 1, 2$

The Production Model

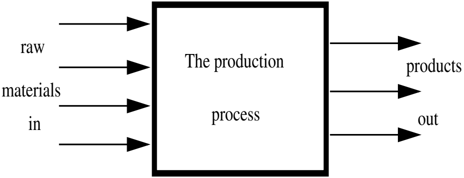


The Production Model



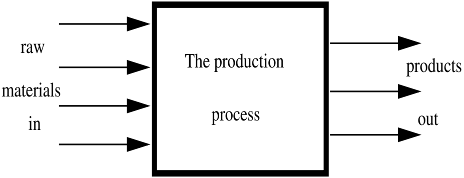
The products are the raw materials reconfigured to look different.

The Production Model



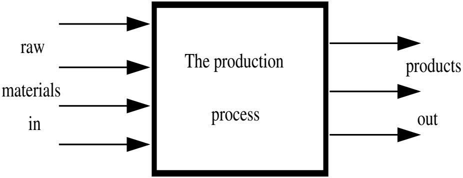
The products are the raw materials reconfigured to look different. Profit is the difference between the purchase price of the raw materials and the sale price of their reconfigured form as products.

The Production Model



The products are the raw materials reconfigured to look different. Profit is the difference between the purchase price of the raw materials and the sale price of their reconfigured form as products. Making a profit means that you sell the raw materials for more than you paid for them.

The Production Model



The products are the raw materials reconfigured to look different.
Profit is the difference between the purchase price of the raw materials and the sale price of their reconfigured form as products.
Making a profit means that you sell the raw materials for more than you paid for them.
On a per unit basis, by how much does the production process increase the value of the raw materials?

Hidden Hand of the Market Place: Duality

In the market place there is competition for raw materials, or the inputs to production. This collective competition is the *hidden hand* that sets the price for goods in the market place.

Hidden Hand of the Market Place: Duality

In the market place there is competition for raw materials, or the inputs to production. This collective competition is the *hidden hand* that sets the price for goods in the market place.

Is there a mathematical model for how these prices are set?

Hidden Hand of the Market Place: Duality

In the market place there is competition for raw materials, or the inputs to production. This collective competition is the *hidden hand* that sets the price for goods in the market place.

Is there a mathematical model for how these prices are set?

Let us think of the market as a separate agent in the market place. It is the agent that owns and sells the raw materials of production.

Hidden Hand of the Market Place: Duality

In the market place there is competition for raw materials, or the inputs to production. This collective competition is the *hidden hand* that sets the price for goods in the market place.

Is there a mathematical model for how these prices are set?

Let us think of the market as a separate agent in the market place. It is the agent that owns and sells the raw materials of production.

The goal of the market is to make the most money possible from its resources by setting the highest prices possible for them.

Hidden Hand of the Market Place: Duality

In the market place there is competition for raw materials, or the inputs to production. This collective competition is the *hidden hand* that sets the price for goods in the market place.

Is there a mathematical model for how these prices are set?

Let us think of the market as a separate agent in the market place. It is the agent that owns and sells the raw materials of production.

The goal of the market is to make the most money possible from its resources by setting the highest prices possible for them.

The market does not want to put the producers out of business, it just wants to take all of their profit.

Hidden Hand of the Market Place: Duality

In the market place there is competition for raw materials, or the inputs to production. This collective competition is the *hidden hand* that sets the price for goods in the market place.

Is there a mathematical model for how these prices are set?

Let us think of the market as a separate agent in the market place. It is the agent that owns and sells the raw materials of production.

The goal of the market is to make the most money possible from its resources by setting the highest prices possible for them.

The market does not want to put the producers out of business, it just wants to take all of their profit.

How can we model this mathematically?

Hidden Hand of the Market Place: Duality

We answer this question in the context of the Plastic Cup Factory.

Hidden Hand of the Market Place: Duality

We answer this question in the context of the Plastic Cup Factory.

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

Hidden Hand of the Market Place: Duality

By how much should the market increase the sale price of plastic resin and hourly labor in order to wipe out the profit for the Plastic Cup Factory?

Hidden Hand of the Market Place: Duality

By how much should the market increase the sale price of plastic resin and hourly labor in order to wipe out the profit for the Plastic Cup Factory?

Define

$$\begin{aligned} 0 \le y_1 &= \text{price increase for a pound of resin} \\ 0 \le y_2 &= \text{price increase for an hour of labor} \end{aligned}$$

Hidden Hand of the Market Place: Duality

By how much should the market increase the sale price of plastic resin and hourly labor in order to wipe out the profit for the Plastic Cup Factory?

Define

$$0 \leq y_1 = \text{price increase for a pound of resin}$$
$$0 \leq y_2 = \text{price increase for an hour of labor}$$

These price increases should wipe out the per unit profitability for cases of both beer mugs and champagne glasses.

Hidden Hand of the Market Place: Duality

production cost increase $\geq$ current profit

Hidden Hand of the Market Place: Duality

$$\text{production cost increase} \geq \text{current profit}$$

Beer Mugs: $\quad \text{cost increase} =$

Hidden Hand of the Market Place: Duality

production cost increase $\geq$ current profit

Beer Mugs: cost increase $= 20y_1 \;+\; \frac{1}{15}y_2$

Hidden Hand of the Market Place: Duality

$$\text{production cost increase} \geq \text{current profit}$$

Beer Mugs: $\quad \text{cost increase} = 20y_1 + \frac{1}{15}y_2 \geq 25$

Hidden Hand of the Market Place: Duality

$$\text{production cost increase} \geq \text{current profit}$$

Beer Mugs: $\text{cost increase} = 20y_1 \; + \; \frac{1}{15}y_2 \; \geq 25$

Champagne Glasses:

Hidden Hand of the Market Place: Duality

production cost increase $\geq$ current profit

Beer Mugs: cost increase $= 20y_1 + \frac{1}{15}y_2 \geq 25$

Champagne Glasses: cost increase $=$

Hidden Hand of the Market Place: Duality

$$\text{production cost increase} \geq \text{current profit}$$

Beer Mugs: cost increase $= 20y_1 + \frac{1}{15}y_2 \geq 25$

Champagne Glasses: cost increase $= 12y_1 + \frac{1}{15}y_2$

Hidden Hand of the Market Place: Duality

production cost increase $\geq$ current profit

Beer Mugs: cost increase $= 20y_1 + \frac{1}{15}y_2 \geq 25$

Champagne Glasses: cost increase $= 12y_1 + \frac{1}{15}y_2 \geq 20$

Hidden Hand of the Market Place: Duality

Now minimize the total increase in the cost of raw materials subject to wiping out the producer's profit. Hopefully this will keep the Plastic Cup Factory in business.

Hidden Hand of the Market Place: Duality

Now minimize the total increase in the cost of raw materials subject to wiping out the producer's profit. Hopefully this will keep the Plastic Cup Factory in business.

$$\text{minimize } 1800y_1 + 8y_2$$

Hidden Hand of the Market Place: Duality

Now minimize the total increase in the cost of raw materials subject to wiping out the producer's profit. Hopefully this will keep the Plastic Cup Factory in business.

$$\text{minimize } 1800y_1 + 8y_2$$

Rewriting the Market's price increase problem we get

Hidden Hand of the Market Place: Duality

Now minimize the total increase in the cost of raw materials subject to wiping out the producer's profit. Hopefully this will keep the Plastic Cup Factory in business.

$$\text{minimize } 1800y_1 + 8y_2$$

Rewriting the Market's price increase problem we get

$$\begin{array}{ll} \text{minimize} & 1800y_1 + 8y_2 \\ \\ \text{subject to} & 20y_1 \; + \; y_2/15 \; \geq 25 \\ & 12y_1 \; + \; y_2/15 \; \geq 20 \\ & 0 \leq y_1, y_2 \end{array}$$

Hidden Hand of the Market Place: Duality

Now minimize the total increase in the cost of raw materials subject to wiping out the producer's profit. Hopefully this will keep the Plastic Cup Factory in business.

$$\text{minimize } 1800y_1 + 8y_2$$

Rewriting the Market's price increase problem we get

$$\begin{array}{ll} \text{minimize} & 1800y_1 + 8y_2 \\ \\ \text{subject to} & 20y_1 \; + \; y_2/15 \; \geq 25 \\ & 12y_1 \; + \; y_2/15 \; \geq 20 \\ & 0 \leq y_1, y_2 \end{array}$$

This is another linear program!

Hidden Hand of the Market Place: Duality

Now minimize the total increase in the cost of raw materials subject to wiping out the producer's profit. Hopefully this will keep the Plastic Cup Factory in business.

$$\text{minimize } 1800y_1 + 8y_2$$

Rewriting the Market's price increase problem we get

$$\begin{array}{ll} \text{minimize} & 1800y_1 + 8y_2 \\ \\ \text{subject to} & 20y_1 \ + \ y_2/15 \ \geq 25 \\ & 12y_1 \ + \ y_2/15 \ \geq 20 \\ & 0 \leq y_1, y_2 \end{array}$$

This is another linear program!

Let us compare this LP with the original LP.

Linear Programming Duality

Linear Programming Duality

Primal:

Linear Programming Duality

Primal: max $25B + 20C$

s.t. $20B + 12C \leq 1800$

$\frac{1}{15}B + \frac{1}{15}C \leq 8$

$0 \leq B, C$

Linear Programming Duality

Primal: max $25B + 20C$

s.t. $20B + 12C \leq 1800$

$\frac{1}{15}B + \frac{1}{15}C \leq 8$

$0 \leq B, C$

Dual:

Linear Programming Duality

Primal:

$$\begin{aligned} \max \quad & 25B + 20C \\ \text{s.t.} \quad & 20B + 12C \leq 1800 \\ & \tfrac{1}{15}B + \tfrac{1}{15}C \leq 8 \\ & 0 \leq B, C \end{aligned}$$

Dual:

$$\begin{aligned} \min \quad & 1800y_1 + 8y_2 \\ \text{s.t.} \quad & 20y_1 + \tfrac{1}{15}y_2 \geq 25 \\ & 12y_1 + \tfrac{1}{15}y_2 \geq 20 \\ & 0 \leq y_1, y_2 \end{aligned}$$

Linear Programming Duality: Matrix Notation

	$\mathcal{P}$			$\mathcal{D}$	
Primal:	max	$c^T x$	**Dual:**	min	$b^T y$
	s.t.	$Ax \leq b$		s.t.	$A^T y \geq c$
		$0 \leq x$			$0 \leq y$

The Weak Duality Theorem of Linear Programming

Theorem: [Weak Duality Theorem]

The Weak Duality Theorem of Linear Programming

Theorem: [Weak Duality Theorem]
If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \leq y^T A x \leq b^T y.$$

The Weak Duality Theorem of Linear Programming

Theorem: [Weak Duality Theorem]
If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \leq y^T A x \leq b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible.

The Weak Duality Theorem of Linear Programming

Theorem: [Weak Duality Theorem]
If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \leq y^T A x \leq b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible.

Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

The Weak Duality Theorem of Linear Programming

Proof:

The Weak Duality Theorem of Linear Programming

Proof:

$$c^T x = \sum_{j=1}^{n} c_j x_j$$

The Weak Duality Theorem of Linear Programming

Proof:

$$
\begin{aligned}
c^T x &= \sum_{j=1}^{n} c_j x_j \\
&\leq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j \quad [0 \leq x_j,\ c_j \leq \sum_{i=1}^{m} a_{ij} y_i \Rightarrow\ c_j x_j \leq (\sum_{i=1}^{m} a_{ij} y_i) x_j]
\end{aligned}
$$

The Weak Duality Theorem of Linear Programming

Proof:

$$\begin{aligned} c^T x &= \sum_{j=1}^{n} c_j x_j \\ &\leq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j \quad [0 \leq x_j,\ c_j \leq \sum_{i=1}^{m} a_{ij} y_i \ \Rightarrow\ c_j x_j \leq (\sum_{i=1}^{m} a_{ij} y_i) x_j] \\ &= y^T A x \end{aligned}$$

The Weak Duality Theorem of Linear Programming

Proof:

$$\begin{aligned} c^T x &= \sum_{j=1}^{n} c_j x_j \\ &\leq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j \quad [0 \leq x_j,\ c_j \leq \sum_{i=1}^{m} a_{ij} y_i \ \Rightarrow \ c_j x_j \leq (\sum_{i=1}^{m} a_{ij} y_i) x_j] \\ &= y^T A x \\ &= \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \end{aligned}$$

The Weak Duality Theorem of Linear Programming

Proof:

$$
\begin{aligned}
c^T x &= \sum_{j=1}^{n} c_j x_j \\
&\leq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j \qquad [0 \leq x_j,\ c_j \leq \sum_{i=1}^{m} a_{ij} y_i \Rightarrow c_j x_j \leq (\sum_{i=1}^{m} a_{ij} y_i) x_j] \\
&= y^T A x \\
&= \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \\
&\leq \sum_{i=1}^{m} b_i y_i \qquad [0 \leq y_i,\ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \Rightarrow (\sum_{j=1}^{n} a_{ij} x_j) y_i \leq b_i y_i]
\end{aligned}
$$

The Weak Duality Theorem of Linear Programming

Proof:

$$
\begin{aligned}
c^T x &= \sum_{j=1}^{n} c_j x_j \\
&\leq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j && [0 \leq x_j,\ c_j \leq \sum_{i=1}^{m} a_{ij} y_i \ \Rightarrow\ c_j x_j \leq (\sum_{i=1}^{m} a_{ij} y_i) x_j] \\
&= y^T A x \\
&= \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \\
&\leq \sum_{i=1}^{m} b_i y_i && [0 \leq y_i,\ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \ \Rightarrow\ (\sum_{j=1}^{n} a_{ij} x_j) y_i \ \leq\ b_i y_i] \\
&= b^T y
\end{aligned}
$$

Test the WDT on the Plastic Cup Factory

,

Test the WDT on the Plastic Cup Factory

$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

,

Test the WDT on the Plastic Cup Factory

$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

Dual:

$$\begin{array}{lrcrl} \min & 1800y_1 + 8y_2 & & & \\ \text{s.t.} & 20y_1 & + & \frac{1}{15}y_2 & \geq 25 \\ & 12y_1 & + & \frac{1}{15}y_2 & \geq 20 \\ & 0 \leq y_1, y_2 & & & \end{array}$$

,

Test the WDT on the Plastic Cup Factory

$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

Dual:

$$\begin{array}{lllll} \min & 1800y_1 + 8y_2 & & & \\ \text{s.t.} & 20y_1 & + & \frac{1}{15}y_2 & \geq 25 \\ & 12y_1 & + & \frac{1}{15}y_2 & \geq 20 \\ & 0 \leq y_1, y_2 & & & \end{array}$$

Dual feasibility of the marginal values:

,

Test the WDT on the Plastic Cup Factory

$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

Dual:

$$\begin{array}{lllll} \min & 1800y_1 + 8y_2 & & & \\ \text{s.t.} & 20y_1 & + & \frac{1}{15}y_2 & \geq 25 \\ & 12y_1 & + & \frac{1}{15}y_2 & \geq 20 \\ & 0 \leq y_1, y_2 & & & \end{array}$$

Dual feasibility of the marginal values:

$$0 \leq \frac{5}{8},$$

Test the WDT on the Plastic Cup Factory

$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

Dual:

$$\begin{array}{lrcrl} \min & 1800y_1 + 8y_2 & & & \\ \text{s.t.} & 20y_1 & + & \frac{1}{15}y_2 & \geq 25 \\ & 12y_1 & + & \frac{1}{15}y_2 & \geq 20 \\ & 0 \leq y_1, y_2 & & & \end{array}$$

Dual feasibility of the marginal values:

$$0 \leq \frac{5}{8},\ 0 \leq \frac{375}{2},$$

Test the WDT on the Plastic Cup Factory

$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

Dual:

$$\begin{array}{lllll} \min & 1800y_1 + 8y_2 & & & \\ \text{s.t.} & 20y_1 & + & \frac{1}{15}y_2 & \geq 25 \\ & 12y_1 & + & \frac{1}{15}y_2 & \geq 20 \\ & 0 \leq y_1, y_2 & & & \end{array}$$

Dual feasibility of the marginal values:

$$0 \leq \frac{5}{8}, \; 0 \leq \frac{375}{2}, \quad 20 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 25,$$

Test the WDT on the Plastic Cup Factory

$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

Dual:

$$\begin{array}{lllll} \min & 1800y_1 + 8y_2 & & & \\ \text{s.t.} & 20y_1 & + & \frac{1}{15}y_2 & \geq 25 \\ & 12y_1 & + & \frac{1}{15}y_2 & \geq 20 \\ & 0 \leq y_1, y_2 & & & \end{array}$$

Dual feasibility of the marginal values:

$$0 \leq \frac{5}{8}, \ 0 \leq \frac{375}{2}, \quad 20 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 25, \quad 12 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 20$$

Test the WDT on the Plastic Cup Factory

$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

Dual:

$$\begin{array}{lllll} \min & 1800y_1 + 8y_2 & & & \\ \text{s.t.} & 20y_1 & + & \frac{1}{15}y_2 & \geq 25 \\ & 12y_1 & + & \frac{1}{15}y_2 & \geq 20 \\ & 0 \leq y_1, y_2 & & & \end{array}$$

Dual feasibility of the marginal values:

$$0 \leq \frac{5}{8},\ 0 \leq \frac{375}{2}, \quad 20 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 25, \quad 12 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 20$$

Equivalence of primal-dual objectives (WDT):

Test the WDT on the Plastic Cup Factory

$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

Dual:

$$\begin{array}{lllll} \min & 1800y_1 + 8y_2 & & & \\ \text{s.t.} & 20y_1 & + & \frac{1}{15}y_2 & \geq 25 \\ & 12y_1 & + & \frac{1}{15}y_2 & \geq 20 \\ & 0 \leq y_1, y_2 & & & \end{array}$$

Dual feasibility of the marginal values:

$$0 \leq \frac{5}{8},\ 0 \leq \frac{375}{2}, \quad 20 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 25, \quad 12 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 20$$

Equivalence of primal-dual objectives (WDT):

$$c^T x = 25 \cdot 45 + 20 \cdot 75 = 2625$$

Test the WDT on the Plastic Cup Factory

$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

Dual:

$$\begin{array}{lllll} \min & 1800y_1 + 8y_2 & & & \\ \text{s.t.} & 20y_1 & + & \frac{1}{15}y_2 & \geq 25 \\ & 12y_1 & + & \frac{1}{15}y_2 & \geq 20 \\ & 0 \leq y_1, y_2 & & & \end{array}$$

Dual feasibility of the marginal values:

$$0 \leq \frac{5}{8}, \; 0 \leq \frac{375}{2}, \quad 20 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 25, \quad 12 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 20$$

Equivalence of primal-dual objectives (WDT):

$$c^T x = 25 \cdot 45 + 20 \cdot 75 = 2625 = 1800 \cdot \frac{5}{8} + 8 \cdot \frac{375}{2} = b^T y$$

What the Weak Duality Theorem Does NOT Say

What the Weak Duality Theorem Does NOT Say

Infeasibility of either $\mathcal{P}$ or $\mathcal{D}$ does not imply the unboundedness of the other.

What the Weak Duality Theorem Does NOT Say

Infeasibility of either $\mathcal{P}$ or $\mathcal{D}$ does not imply the unboundedness of the other.

It is possible for **both** $\mathcal{P}$ and $\mathcal{D}$ to be infeasible.

What the Weak Duality Theorem Does NOT Say

Infeasibility of either $\mathcal{P}$ or $\mathcal{D}$ does not imply the unboundedness of the other.

It is possible for **both** $\mathcal{P}$ and $\mathcal{D}$ to be infeasible.

Example:

$$\begin{array}{rrcrcr} \text{maximize} & 2x_1 & - & x_2 & & \\ & x_1 & - & x_2 & \leq & 1 \\ & -x_1 & + & x_2 & \leq & -2 \\ & 0 & \leq & x_1, & x_2 & \end{array}$$