Winter 2008: MATH 582G.
Convex Optimization.
Project: Problem 6.14

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Interpolation with positive-real functions.

Suppose $z_1, \ldots, z_n \in \mathbb{C}$ are n distinct points with $|z_i| > 1$. We define K_{np} as the set of vectors $y \in \mathbb{C}^n$ for which there exists a function $f : \mathbb{C} \to \mathbb{C}$ that satisfies the following conditions:

- f is *positive-real*, which means that it is analytic outside the unit ball and its real part is nonnegative outside the unit ball (i.e. for $|z| > 1$).
- f satisfies the *interpolation conditions*: $f(z_i) = y_i$ for all i.

We denote the set of all positive-real functions as $\mathcal{F}$.

Question (a) (Show that K_{np} is closed convex cone.). If vectors y_1 and y_2 can be represented via positive-real functions f_1 and f_2 then for $a > 0$ vector $y_1 + ay_2$ corresponds to function $f_1 + af_2$ which is also positive-real. Thus K_{np} is a cone.

To prove that this cone is closed we need some results from harmonic functions theory (we use [1] as a reference). Let us denote by $h^1(B)$ the harmonic Hardy space (B is the unit ball $|z| < 1$):

$$h^1(B) \equiv \{u \,|\, u \text{ harmonic}, \|u\|_{h^1} < \infty\}$$
$$\|u\|_{h^1} = \sup_{0 \le r < 1} \|u_r\|_1$$

where u_r is the restriction of u on the sphere $|z| = r$. For any complex Borel measure μ on the unit sphere $S = \{|z| = 1\}$ we can define the Poisson integral:

$$P[\mu] = \int_S \Re\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) d\mu(\theta) = \int_S \left(\frac{1 - |z|^2}{|e^{i\theta} - z|^2}\right) d\mu(\theta)$$

which is harmonic function on the ball B (since we can differentiate under the integral sign).

Theorem 6.13 ([1]) states that the map $\mu \to P[\mu]$ is a linear isometry from the space of complex Borel measures $M(S)$ on the sphere S with the finite total variation norm and the Hardy space $h^1(B)$ with the norm $\|\dot{}\|_{h^1}$. In particular, computing residuals it is easy to show that if harmonic function $u(z)$ is continuous on $\bar{B}$ then the corresponding Borel measure is $u(e^{i\theta})d\lambda(\theta)$ where $d\lambda(\theta)$ is the uniform measure. Finally, Corollary 6.15 states that if u is nonnegative on B then $\|u\|_{h^1} = u(0)$ and there is a unique nonengative Borel measure μ such that $P[\mu] = u$. Converse is obviously true since Poisson kernel:

$$P(e^{i\theta}, z) = \Re\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) = \frac{1 - |z|^2}{|e^{i\theta} - z|^2}$$

is non-negative on B.

The space $M(S)$ is isometrically isomorphic to the dual space $C(S)^*$ and since $C(S)$ is separable Banach space any bounded sequence of measures μ_n contains weak-* converging subsequence μ_{n_k}. In particular, this implies pointwise convergence $\lim_n P[\mu_{n_k}](z) = P[\mu_\infty](z)$ for all $z \in B$ where μ_∞ is the weak-* limit.

Now, let f be positive-real. Then $f(z^{-1})$ is analytic in B, $u(z) = \Re f(z^{-1})$ is nonnegative harmonic and from the argument above we obtain that there exists nonnegative Borel measure μ such that $u = P[\mu]$. But the analytic function is determined by its real part up to an imaginary constant. Since the real part of the analytic function:

$$\int_S \left(\frac{e^{i\theta} + z}{e^{i\theta} - z} \right) d\mu(\theta)$$

is equal to $u(z)$ we have that

$$f(z) = i\Im f(\infty) + \int_S \left(\frac{e^{i\theta} + z^{-1}}{e^{i\theta} - z^{-1}} \right) d\mu(\theta).$$

Thus there is one to one correspondence between $\mathcal{F}$ and $M(S) \times \mathbb{R}$. Moreover, total variation of measure $\|\mu\|$ is equal to $u(0) = \Re f(\infty)$.

There exists a holomorphic automorphism $H(z; \zeta)$ of the unit ball such that $H(\zeta; \zeta) = 0$:

$$H(z; \zeta) = \frac{z - \zeta}{1 - \bar{\zeta} z}$$

The inverse mapping for $H(z; \zeta)$ is defined by $H(z; -\zeta)$. Therefore, $\tilde{H}(z) = H(z^{-1}; -z_1^{-1})^{-1}$ is an automorphism of the compliment to the unit ball which maps ∞ to z_1. Function $f(\tilde{H}(z))$ is real-positive so considering it instead of $f(z^{-1})$ we obtain that without loss of generality we can assume $z_1 = \infty$.

Finally, if f_n are positive-real functions which represent vectors y_n for $z_1 = \infty, z_2, \ldots, z_n$ and $y_n \to y$ for some vector y then for corresponding Borel measures μ_n we have $\|\mu_n\| = \Re f_n(\infty) = \Re y_n \to \Re y$ thus the sequence μ_n is bounded in norm and contains weak-* converging subsequence μ_{n_k}. The weak-* limit μ_∞ of this subsequence is nonengative Borel measure such that $\lim_{n_k} P[\mu_{n_k}](z) = P[\mu_\infty](z)$ for every z. Thus y is representable by positive-real function which corresponds to $P[\mu_\infty]$ and the cone K_{np} is closed.

Question (b) (Find the dual cone K_{np}^*.). The inner product on $\mathbb{C}^n$ is defined by $\Re(x^H y) = \Re(y^H x)$ where y^H is the complex conjugate transpose (this inner product corresponds to the usual Eucleadean structure on the corresponding real space). By definition $x \in K_{np}^*$ if for all $y \in K_{np}$ we have $\Re(y^H x) \geq 0$. Or equivalently,

$$\begin{aligned} &- \sum_k \Re(x_k(iA)) + \int_S \Re \left(\sum_k x_k \frac{e^{-i\theta} + \bar{z}_k^{-1}}{e^{-i\theta} - \bar{z}_k^{-1}} \right) d\mu(\theta) = \\ &= A\Im(\mathbf{1}^T x) + \int_S \Re \left(\sum_k x_k \frac{e^{-i\theta} + \bar{z}_k^{-1}}{e^{-i\theta} - \bar{z}_k^{-1}} \right) d\mu(\theta) \geq 0 \end{aligned}$$

for any nonnegative Borel measure μ and any $A \in \mathbb{R}$. For $\mu = 0$ and $A = \pm 1$ we obtain:

$$\pm \Im(\mathbf{1}^T x) \geq 0 \Rightarrow \Im(\mathbf{1}^T x) = 0.$$

Since the function

$$T(\theta) = \Re \left(\sum_k x_k \frac{e^{-i\theta} + \bar{z}_k^{-1}}{e^{-i\theta} - \bar{z}_k^{-1}} \right)$$

is continuous in θ it should be nonnegative. Otherwise, there exists the open interval I where it is less then $\varepsilon < 0$ and the uniform measure $d\lambda_I$ on this interval would contradict the assumption:

$$\int_S T(\theta) d\lambda_I(\theta) = \int_I T(\theta) d\lambda_I(\theta) \leq \varepsilon < 0$$

for $A = 0$. On the other hand, if $\Im(\mathbf{1}^T x = 0)$ and $T(\theta) \geq 0$ for all $\theta \in S$ then $\Re(y^H x) \geq 0$ for all $y \in K_{np}$. Thus:

$$K_{np}^* = \left\{ x \in \mathbb{C}^n \,\middle|\, \Im(\mathbf{1}^T x) = 0, \Re\left(\sum_k x_k \frac{e^{-i\theta} + \bar{z}_k^{-1}}{e^{-i\theta} - \bar{z}_k^{-1}}\right) \geq 0, \forall \theta \in S \right\}$$

Question (c) (Simplify the definition of the cone K_{np}^*.). Let us introduce the following notation:

$$v_k(\theta) = \frac{e^{i\theta} + z_k^{-1}}{e^{i\theta} - z_k^{-1}}$$
$$w_k(\theta) = \frac{1}{e^{i\theta} - z_k^{-1}}$$

Then we have:

$$w_k(\theta)\bar{w}_l(\theta) = \frac{1}{2(1 - z_k^{-1}\bar{z}_l^{-1})}(v_k(\theta) + \bar{v}_l(\theta))$$

If $x \in K_{np}^*$ then:

$$H(\theta; x) = \Re\left(\sum_k x_k \bar{v}_k(\theta)\right) = \frac{1}{2}\left(\sum_k x_k \bar{v}_k(\theta) + \bar{x}_k v_k(\theta)\right) \geq 0, \forall \theta \in S$$

The representation above is unique in a sense that for $x \neq x'$ we have $H(\theta; x) \neq H(\theta; x')$ for some θ. Indeed, we have:

$$H(\theta; x) = \Re\left(\frac{q(e^{i\theta}; x)}{p(e^{i\theta})}\right)$$
$$p(z) \equiv \prod_k (z - z_k^{-1})$$

where $q(z; x)$ is a polynomial such that $\deg q(z; x) \leq \deg p(z)$. Function $q(z; x)/p(z)$ is analytical outside the unit ball and therefore is uniquely determined by its real part on the unit sphere. Thus $q(z; x) = q(z; x')$ and $x = x'$.

From the definition of K_{np}^* we have $\Im(\mathbf{1}^T x) = 0$ and:

$$\sum_k x_k = \Re(\mathbf{1}^T x) + i\Im(\mathbf{1}^T x) = \Re(\mathbf{1}^T x) = \Re(\mathbf{1}^T x) - i\Im(\mathbf{1}^T x) = \sum_k \bar{x}_k$$

Therefore $H(\theta; x)$ belongs to the linear subspace generated by the set $\{v_k + \bar{v}_l\}$ and thus there exist such matrix H_{kl} that:

$$H(\theta; x) = \sum_{k,l} H_{kl} w_k(\theta) \bar{w}_l(\theta)$$

Since $H(\theta; x) \in \mathbb{R}$ we have:

$$H(\theta; x) = \frac{1}{2}(H(\theta; x) + \bar{H}(\theta; x)) = \left(\sum_{k,l} H_{kl} w_k(\theta) \bar{w}_l(\theta) + \sum_{k,l} \bar{H}_{kl} \bar{w}_k(\theta) w_l(\theta) \right) =$$

$$= \sum_{k,l} \frac{H_{kl} + \bar{H}_{lk}}{2} w_k(\theta) \bar{w}_l(\theta)$$

and thus we can assume that $H_{kl} = \bar{H}_{lk}$ i.e. matrix H_{kl} is Hermitian. Since $H(\theta; x)$ is uniquely defined by x we have:

$$H(\theta; x) = \sum_{k,l} H_{kl} w_k(\theta) \bar{w}_l(\theta) = \frac{1}{2} \sum_{k,l} \frac{H_{kl}}{1 - \bar{z}_k^{-1} \bar{z}_l^{-1}} (v_k + \bar{v}_l) =$$

$$= \Re \left(\sum_k \left(\sum_l \frac{H_{kl}}{1 - z_k^{-1} \bar{z}_l^{-1}} \right) v_k \right) \Rightarrow \bar{x}_k = \sum_l \frac{H_{kl}}{1 - z_k^{-1} \bar{z}_l^{-1}} \Rightarrow x_k = \sum_k \frac{H_{kl}}{1 - z_k^{-1} \bar{z}_l^{-1}}$$

Since every Hermitian bilinear form can be diagonalized, for Hermitian matrix H_{kl} there exist such vectors $h^1, \dots, h^n$ that

$$H = \sum_k \pm h^k (h^k)^H$$

for certain choice of signs. Therefore, we have:

$$H(\theta; x) = \sum_{m,l} H_{kl} w_m(\theta) \bar{w}_l(\theta) = \sum_k \pm \sum_{m,l} h_m^k \bar{h}_l^k w_m(\theta) \bar{w}_l(\theta) = \sum_k \pm \left| \sum_l h_l^k w_l \right|^2.$$

We can notice that for positive semidefinite matrix H_{kl} all signs are positive and therefore corresponding function $H(\theta; x)$ is nonnegative. On the other hand, for an arbitrary Hermitian H_{kl} we have

$$H(\theta; x) == \frac{1}{|p(e^{i\theta})|^2} \sum_{k=0}^{n} (q_k e^{ik\theta} + \bar{q}_k e^{-ik\theta})$$

for some q_k. From Riesz-Fejér theorem it follows that $H(\theta; x) \geq 0$ implies the existance of such p_k that

$$H(\theta; x) = \frac{1}{|p(e^{i\theta})|^2} \left| \sum_{k=0}^{n} p_k e^{ik\theta} \right|^2.$$

and therefore $H(\theta; x)$ can be represented by Hermitian positive semidefinite matrix H_{kl}. Thus we have:

$$K_{np}^* = \left\{ x \in \mathbb{C}^n \middle| \exists Q \in H_+^n, x_l = \sum_k \frac{Q_{kl}}{1 - z_k^{-1} \bar{z}_l^{-1}} \right\}$$

Question (d) (Find the cone K_{np}.). Let us consider the cone:

$$L \equiv \left\{ x \in \mathbb{C}^n \middle| \exists h \in \mathbb{C}^n, x_k = \sum_l \frac{h_k \bar{h}_l}{1 - \bar{z}_k^{-1} z_l^{-1}} \right\}$$

Since every Hermitian positive semidefinite matrix can be represented as a sum of $h_k(h_k)^H$ for certain vectors h_k and any such sum is Hermitian positive semidefinite matrix we have $K_{np}^* = \mathrm{conv}(L)$. The cone K_{np} is closed, thus:

$$K_{np} = K_{np}^{**} = (\mathrm{conv}(L))^* = L^* = \left\{ y \in \mathbb{C}^n \,\middle|\, \Re(x^H y) \geq 0, x \in L \right\}$$

For $x \in L$ we have:

$$2\Re(y^H x) = y^H x + x^H y = \sum_{k,l} \frac{\bar{y}_l h_k \bar{h}_l}{1 - z_k^{-1} \bar{z}_l^{-1}} + \sum_{k,l} \frac{y_l \bar{h}_k h_l}{1 - \bar{z}_k^{-1} z_l^{-1}} =$$

$$= \sum_{k,l} \frac{(y_k + \bar{y}_l) h_k \bar{h}_l}{1 - z_k^{-1} \bar{z}_l^{-1}} = h^H P(y) h$$

$$P(y)_{kl} \equiv \frac{y_k + \bar{y}_l}{1 - z_k^{-1} \bar{z}_l^{-1}}$$

Thus $y \in K_{np}$ if and only if the matrix $P(y)$ is Hermitian positive semidefinite:

$$h^H P(y) h \geq 0, \quad \forall h \in \mathbb{C}^n$$

Matrix $P(y)$ is called *Nevanlinna-Pick matrix* associated with points z_k, y_k.

Question (e) (Approximation problem). The problem of approximation with positive-real functions:

$$\min_{f \in \mathcal{F}} \sum_{k=1}^{n} \|y_i - f(z_i)\|^2$$

is equivalent to finding the distance from y to the set K_{np}:

$$\inf_{u \in K_{np}} \|y - u\|^2 = dist(y|K_{np})$$

Since K_{np} is closed convex cone this problem is convex. Moreover, we can write

$$d(y|K_{np}) = (\sigma_{\mathbb{B}^\circ} \Box \delta_{K_{np}})(z) = (\delta_{\mathbb{B}^\circ} + \sigma_{K_{np}})^*(z) =$$
$$= (\delta_{\mathbb{B}^\circ} + \delta_{K_{np}^\circ})^*(z) = \sigma_{\mathbb{B}^\circ \cap K_{np}^\circ}(z)$$

which means that the distance function is the support function to the convex set $\mathbb{B}^\circ \cap K_{np}^\circ$.

References

[1] Sheldon Axler, Paul Bourdon, and Wade Ramey. *Harmonic function theory*, volume 137 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001.