1 Chapter 5.19

1.1 (a)

Given a vector $x \in \mathbb{R}^n$, show that $f(x)$ is equal to the optimal value of the LP

$$\begin{array}{rl} maximize & x^T y \\ subject\ to & 0 \leq y \leq \mathbf{1} \\ & \mathbf{1}^T y = r \end{array}$$

with $y \in \mathbb{R}^n$ as variable.

1.2 Ans:

It's obvious that $f(x)$ is equivalent to an optimization problem as follow:

$$\begin{array}{rl} maximize & x^T y \\ subject\ to & y_i = 0\ or\ 1, \quad i = 1, ..., n \\ & \mathbf{1}^T y = r \end{array}$$

And this problem is equivalent to the one given in (a), since we can always rearrange the balance of y_i, meaning assigning more weight(actually 1) to the y_is that correspond to greater x_is.

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2.1 (b)

Derive the dual of the LP in part (a). Show that it can be written as

$$\begin{array}{rl} minimize & rt + \mathbf{1}^T u \\ subject\ to & t\mathbf{1} + u \geq x \\ & u \geq 0 \end{array}$$

where the variables are $t \in \mathbb{R}, u \in \mathbb{R}^n$. By duality this LP has the same optimal value as the LP in (a), i.e., $f(x)$. We therefore have the following result: x satisfies $f(x) \leq \alpha$ if and only if there exist $t \in \mathbb{R}, u \in \mathbb{R}^n$ such that

$$rt + \mathbf{1}^T u \leq \alpha, \quad t\mathbf{1} + u \geq x, \quad u \geq 0$$

2.2 Ans:

$$\begin{array}{rcl} L(y, v, u, t) & = & -x^T y - v^T y + u^T(y - \mathbf{1}) + t(\mathbf{1}^T y - r) + \delta_{\mathbb{R}^n_+}(v, u) \\ g(v, u, r) & = & \inf_y (-x - v + u + t\mathbf{1})^T y - u^T \mathbf{1} - rt + \delta_{\mathbb{R}^n_+}(v, u) \\ & = & \begin{cases} -u^T \mathbf{1} - rt, & if \quad u + t\mathbf{1} \geq x, u \in \mathbb{R}^n_+; \\ -\infty, & otherwise. \end{cases} \end{array}$$

This proves that the dual can be reformulated as shown in part (b).

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3.1 (c)

As an application, we consider an extension of the classical Markowitz portfolio optimization problem

$$\begin{array}{ll} \textit{minimize} & x^T \Sigma x \\ \textit{subject to} & \bar{p}^T x \geq r_{min} \\ & \mathbf{1}^T x = 1, \quad x \geq 0 \end{array}$$

discussd in chapter 4, page 155. THe avriable is the portfolio $x \in \mathbb{R}^n; barp$ and Σ are the mean and covariance matrix of the price change vector p.

Suppose we add a diversification constraint, requiring that no more than 80% of the total budget can be invested in any 10% of the assets. This constranit can be expressed as

$$\sum_{i=1}^{\lfloor 0.1n \rfloor} x_{[i]} \leq 0.8$$

Formulate the portfolio optimization problem with diversification constraint as a QP.

3.2 Classical Markowitz portfolio optimization problem (single period)

parameters:

n number of assets or stocks.

x_i amount of asset i held throughout the period,
$x_i > 0$ means long position, $x_i < 0$ means short position.

p_i relative price change of asset i – a random variable. Their mean is given by $\bar{p}$, covariance matrix Σ. One p_i can be a constant, representing a risk-free asset.

$P^T x$ overall return

variables:

x allocation of different assets

constraints:

$x \geq 0$ No short positions.

$\mathbf{1}^T x = 1$ Total budget is 1.

$\overline{r} = \overline{p}^T x \geq r_{min}$ minimum acceptable mean return.

object function:

$x^T \Sigma x$ return variance, which is positively related to risk (volatility) of the portfolio.

3.3

Define $r \triangleq \llcorner 0.1n \lrcorner$. According to (a) and (b), the diversification constraint can be written as:

$$u^T \mathbf{1} + rt \leq 0.8, \quad u + t\mathbf{1} \geq x, \quad u \geq 0$$

Then the modified Markowitz portfolio optimization problem becomes

$$\begin{array}{rc} minimize & x^T \Sigma x \\ subject\ to & \overline{p}^T x \geq r_{min} \\ & u^T \mathbf{1} + rt \leq 0.8 \\ & u + t\mathbf{1} \geq x \\ & \mathbf{1}^T x = 1 \\ & x \geq 0 \\ & u \geq 0 \end{array}$$

3.4 Dual of modified Markowitz problem

$$\begin{aligned} L(x, u, t, m, s, \lambda, \mu, \xi, z) &= x^T \Sigma x + m(r_{min} - \overline{p}^T x) + s(u^T \mathbf{1} + rt - 0.8) \\ &+ \lambda^T (x - u - t\mathbf{1}) + \mu(\mathbf{1}^T x - 1) - \xi^T x - z^T u \\ &+ \delta_{\mathbb{R}^n_+}(\lambda, \xi, z) + \delta_{\mathbb{R}_+}(m, s) \end{aligned}$$

$$\begin{aligned} g(m, s, \lambda, \mu, \xi, z) &= \inf_{x,u,t} L(x, u, t, m, s, \lambda, \mu, \xi, z) \\ &= \inf_{x} x^T \Sigma x - (m\overline{p} + \xi - \lambda - \mu\mathbf{1})^T x \\ &+ \inf_{u} (s\mathbf{1} - \lambda - z)^T u \\ &+ \inf_{t} (sr - \mathbf{1}^T \lambda) t \\ &+ m r_{min} - 0.8s - \mu + \delta_{\mathbb{R}^n_+}(\lambda, \xi, z) + \delta_{\mathbb{R}_+}(m, s) \end{aligned}$$

Here, we assume Σ is positive definite. Then for the first part, we take the first derivative w.r.t. x and set it to be 0:

$$2\Sigma x - (m\overline{p} + \xi - \lambda - \mu\mathbf{1}) = 0$$

Denote $\sigma = m\overline{p} + \xi - \lambda - \mu\mathbf{1}$, then we have

$$x = \frac{1}{2}\Sigma^{-1}\sigma$$
$$\inf_x x^T\Sigma x - (m\overline{p} + \xi - \lambda - \mu\mathbf{1})^T x = -\frac{1}{4}\sigma^T\Sigma^{-1}\sigma$$

So,

$$g(m, s, \lambda, \mu, \xi, z) = \begin{cases} -\frac{1}{4}\sigma^T\Sigma^{-1}\sigma, & if \quad x = \frac{1}{2}\Sigma^{-1}\sigma, \quad \lambda, \xi, z \geq 0, \quad m, s \geq 0 \\ & s\mathbf{1} - \lambda - z = 0, \quad sr - \mathbf{1}^T\lambda = 0 \\ -\infty, & otherwise. \end{cases}$$

Finally, the dual problem is

$$\begin{array}{rc} maximize & -\frac{1}{4}\sigma^T\Sigma^{-1}\sigma \\ subject\ to & x = \frac{1}{2}\Sigma^{-1}\sigma \\ & \sigma = m\overline{p} + \xi - \lambda - \mu\mathbf{1} \\ & s\mathbf{1} - \lambda - z = 0 \\ & sr - \mathbf{1}^T\lambda = 0 \\ & \lambda, \xi, z \geq 0 \\ & m, s \geq 0 \end{array}$$