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Problem 4.62

Optimal power and bandwidth allocation in a Gaussian broadcast channel. We consider a communication system in which a central node transmits messages to n receivers. ('Gaussian' refers to the type of noise that corrupts the transmissions.) Each receiver channel is characterized by its (transmit) power level $P_i \geq 0$ and its bandwidth $W_i \geq 0$. The power and bandwidth of a receiver channel determine its *bit rate* R_i (the rate at which information can be sent) via

$$R_i = \alpha_i W_i \log(1 + \beta_i P_i / W_i),$$

where α_i and β_i are known positive constants. For $W_i = 0$, we take $R_i = 0$ (which is what you get if you take the limit as $W_i \to 0$).

The powers must satisfy a total power constraint, which has the form

$$P_1 + \cdots + P_n = P_{\text{tot}},$$

where $P_{\text{tot}} > 0$ is a given total power available to allocate among the channels. Similarly, the bandwidths must satisfy

$$W_1 + \cdots + W_n = W_{\text{tot}},$$

where $T_{\text{tot}} > 0$ is the (given) total available bandwidth. The optimization variables in this problem are the powers and the bandwidths, that is, $P_1, \ldots, P_n, W_1, \ldots, W_n$.

The objective is to maximize the total utility,

$$\sum_{i=1}^{n} u_i(R_i),$$

where $u_i : \mathbb{R} \to \mathbb{R}$ is the utility function associated with the ith receiver. (You can think of $u_i(R_i)$ as the revenue obtained for providing a bit rate R_i to receiver i, so the objective is to maximize the total revenue.) You can assume that the utility functions u_i are nondecreasing and concave.

Pose this problem as a convex optimization problem.

This is easy; just change the maximization to minimizing the negative of the total utility. Also, write the constraints in matrix form. We have:

$$\begin{aligned}
\underset{P,W}{\text{minimize}} \quad & \sum_{i=1}^{n} -u_i\left(\alpha_i W_i \log\left(1 + \frac{\beta_i P_i}{W_i}\right)\right) \qquad (\mathcal{P}) \\
\text{subject to} \quad & \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}^T \begin{bmatrix} P \\ W \end{bmatrix} - P_{\text{tot}} = 0 \\
& \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}^T \begin{bmatrix} P \\ W \end{bmatrix} - W_{\text{tot}} = 0 \\
& P \succeq \mathbf{0} \\
& W \succeq \mathbf{0}
\end{aligned}$$

where $P = \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix}^T$, $W = \begin{bmatrix} W_1 & W_2 & \cdots & W_n \end{bmatrix}^T$, and $\mathbf{1}$ and $\mathbf{0}$ are vectors of size $n \times 1$.

The constraints are affine; all we have to do is show that the objective function is actually convex.

Define $g(P_i) = \log(1 + \beta_i P_i)$ Note that g is clearly concave. (One way to see this is that $1 + \beta_i P_i$ is affine (so concave) in P_i, and log is concave and increasing, so the concavity of g follows from composition rules.) Now, recall that g is concave if and only if the perspective of g is concave. Note that the perspective of g is exactly

$$W_i g(P_i/W_i) = W_i \log(1 + \beta_i P_i/W_i)$$

Since this function is concave, we can multiply by the postive scalar α_i to see that

$$\alpha_i W_i \log\left(1 + \frac{\beta_i P_i}{W_i}\right)$$

is concave. Since u_i is nondecreasing and concave, the composition $-u_i\left(\alpha_i W_i \log\left(1 + \beta_i P_i/W_i\right)\right)$ is concave. Taking the negative of this gives

$$-u_i\left(\alpha_i W_i \log\left(1 + \frac{\beta_i P_i}{W_i}\right)\right)$$

is convex. Thus the objective is the sum of convex functions, so is convex.

Now we have our convex problem. Next we'll find the dual problem.

For convenience, let

$$f_0(P, W) = \sum_{i=1}^{n} -u_i\left(\alpha_i W_i \log\left(1 + \frac{\beta_i P_i}{W_i}\right)\right)$$

Then the Lagrangian is:

$$\begin{aligned} L(P, W, \lambda_1, \lambda_2, \mu_1, \mu_2) &= f_0(P, W) + \lambda_1\left(\begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix}^T \begin{bmatrix} P \\ W \end{bmatrix} - P_{\text{tot}}\right) + \lambda_2\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}^T \begin{bmatrix} P \\ W \end{bmatrix} - W_{\text{tot}}\right) - \mu_1^T P - \mu_2^T W \\ &\quad \text{where } \mu_1, \mu_2 \succeq 0 \\ &= -\lambda_1 P_{\text{tot}} - \lambda_2 W_{\text{tot}} + f_0(P, W) + \left(\begin{bmatrix} \mathbf{1}_{1\times n} & \mathbf{0}_{1\times n} & -I_n & \mathbf{0}_{n\times n} \\ \mathbf{0}_{1\times n} & \mathbf{1}_{1\times n} & \mathbf{0}_{n\times n} & -I_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \mu_1 \\ \mu_2 \end{bmatrix}\right)^T \begin{bmatrix} P \\ W \end{bmatrix} \\ &\quad + \delta_{\mathbb{R}_+^n}(\mu_1) + \delta_{\mathbb{R}_+^n}(\mu_1). \end{aligned}$$

The dual problem is then

$$\underset{\lambda, \mu}{\text{maximize}} \quad \inf_{P, W} L(P, W, \lambda_1, \lambda_2, \mu_1, \mu_2)$$

Let's see if we can make the formulation a bit more explicit. The infemum is independent of the first few terms of L, so we can pull them out. Beyond this, we can change $\inf x$ to $-\sup\{-x\}$ so as to write the dual in terms of a conjugate function. Finally, the indicator functions change back to ordinary constraints.

$$
\begin{aligned}
\underset{\lambda,\mu}{\text{maximize}} \quad & -\lambda_1 P_{\text{tot}} - \lambda_2 W_{\text{tot}} + f_0^* \left(\begin{bmatrix} -\mathbf{1}_{1\times n} & \mathbf{0}_{1\times n} & I_n & \mathbf{0}_{n\times n} \\ \mathbf{0}_{1\times n} & -\mathbf{1}_{1\times n} & \mathbf{0}_{n\times n} & I_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \mu_1 \\ \mu_2 \end{bmatrix} \right) \\
\text{subject to} \quad & \mu_1, \mu_2 \succeq 0.
\end{aligned} \tag{$\mathcal{D}$}
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