AMath 582
Project
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Problem 4.60

Investment period $t = 1, 2 \cdots, T$
Asset $i = 1, 2, \cdots, n$
Allocation strategy $x \in \mathbf{R}^n_+, \mathbf{1}^T x = 1$; fixed for each period such that $x_i(t)$=the amount of money that's invested in asset i in period t and the total money available for all assets is normalized to 1.
Wealth at the beginning of the period $t = W(t)$ and the return rate during the period t is $\lambda(t)$ such that $W(t+1) = \lambda(t)W(t)$
Total return rate= $\frac{W(N)}{W(0)} = \prod_{t=1}^T \lambda(t)$
Growth rate of the investment over the N periods $\triangleq \frac{1}{N} \log \prod_{t=1}^T \lambda(t) = \frac{1}{N} \sum_{t=1}^T \log \lambda(t)$
We're interested in determining an allocation strategy x that maximizes the growth rate for a long term i.e. large N. There is uncertainty in the return rate and we use a discrete stochastic model to account for the uncertainty in the returns. For each period t, the identical independent distribution is assumed for the random varible $\lambda(t)$.
Possible senarios $j = 1, 2, \cdots, m$
The return for asset i under j senario $= p_{ij}$ and P is a matrix that has an entry p_{ij} such that $P \in \mathbf{R}^{\mathbf{n} \times \mathbf{m}}$. $P_j = j^{th}$ column in P.
The probability of the total return under j senario $= \pi_j = \Pr(\lambda(t) = P_j^T x)$
Recall the law of large numbers.
$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^T \log \lambda(t) = \mathbf{E}(\log(\lambda(t))) = \sum_{j=1}^m \pi_j \log(P_j^T x)$
Then we have a following optimization problem.

$$\begin{array}{c} \text{maximize } \sum_{j=1}^m \pi_j \log(P_j^T x) \\ \text{subject to } x \geq 0 \text{ and } \mathbf{1}^T x = 1 \end{array}$$

$\log(P_j^T x)$ is concave in x on its domain $\{x | P_j^T x > 0\}$. Since $\pi_j \geq 0 \ \forall j, \ \pi_j \log(P_j^T x)$ is also convave. The sum of concave functions is also concave so we have maximization of the concave objective function suject to convex constraints. Therefore, this is a convex optimization.
Rewrite the problem as

$$\begin{array}{c} \text{minimize } -\sum_{j=1}^m \pi_j \log(y_j) \\ \text{subject to } y_j = P_j^T x \ \forall j, \ x \geq 0, \ \mathbf{1}^T x = 1 \end{array}$$

Introduce dual variables μ, ν, γ for each constraints. Then $\mu \in \mathbf{R}^m, \nu \in \mathbf{R}^n_+, \gamma \in \mathbf{R}$.

$$
\begin{aligned}
L(x,y,\mu,\nu,\gamma) &= -\sum_{j=1}^{m}\pi_j\log(y_j)+\mu^T(y-Px)-\nu^Tx+\gamma(\mathbf{1}^Tx-1)-\delta_{\mathbf{R}^n_+}(\nu) && (1)\\
&= -\sum_{j=1}^{m}\pi_j\log(y_j)+\mu^Ty+(\gamma\mathbf{1}^T-\mu^TP^T-\nu^T)x-\gamma-\delta_{\mathbf{R}^n_+}(\nu) && (2)\\
g(\mu,\nu,\gamma) &= \inf_{x,y} L && (3)\\
&= -\gamma+\inf_y(-\sum_{j=1}^{m}\pi_j\log(y_j)+\mu^Ty) && (4)\\
& \quad \text{provided } \gamma\mathbf{1}^T-\mu^TP^T-\nu^T=0 \text{ i.e } \gamma\mathbf{1}-P\mu=\nu\geq 0 && (5)\\
& \quad -\frac{\pi_j}{y_j}+\mu_j=0 \ \Rightarrow y_j=\frac{\pi_j}{\mu_j} && (6)\\
g(\mu,\gamma) &= 1-\gamma+\sum_{j=1}^{m}\pi_j\log(\frac{\mu_j}{\pi_j}) && (7)\\
& \quad \text{provided } \gamma\mathbf{1}\geq P\mu && (8)\\
& && (9)
\end{aligned}
$$

Let $\frac{\mu}{\gamma}=\tilde{\mu}$ since $\mu>0$ and $\gamma>0$. Then,

$$
\begin{aligned}
\sum_{j=1}^{m}\pi_j\log(\frac{\mu_j}{\pi_j}) &= \sum_{j=1}^{m}\pi_j\log(\frac{\gamma\tilde{\mu}}{\pi_j}) && (10)\\
&= \log\gamma+\sum_{j=1}^{m}\pi_j\log(\frac{\tilde{\mu}}{\pi_j}) && (11)\\
\gamma\mathbf{1}\geq P\mu &\Rightarrow \mathbf{1}\geq P\tilde{\mu} && (12)\\
\sup_{\tilde{\mu},\gamma} g(\tilde{\mu},\gamma) &= \sup_{\tilde{\mu},\gamma}(1-\gamma+\log\gamma+\sum_{j=1}^{m}\pi_j\log(\frac{\tilde{\mu}}{\pi_j}) && (13)\\
&= \sup_{\tilde{\mu}}\sum_{j=1}^{m}\pi_j\log(\frac{\tilde{\mu}}{\pi_j}) && (14)\\
& && (15)
\end{aligned}
$$

Dual problem: maximize $\sum_{j=1}^{m}\pi_j\log(\frac{\tilde{\mu}}{\pi_j})$
subject to $\mathbf{1}\geq P\tilde{\mu}$