Qiuying Lin
03/13/2008

Problem 4.59

Proof. (a)For the stochastic optimization problem

$$\begin{array}{ll} \min_{x \in \mathbf{R}^n} & \mathbf{E} f_0(x,u) \\ \text{s.t.} & \mathbf{E} f_i(x,u) \leq 0, \quad i = 1,\dots,m \end{array}$$

where the expectation is with respect to $u \in \mathbf{R}^k$, and $f_i, i = 0,\dots,m$ are convex in $x \in \mathbf{R}^n$ for each u. For $\forall x_1, x_2 \in \mathbf{R}^n, \forall \lambda \in [0,1]$, we know

$$\begin{aligned} f_i(\lambda x_1 + (1-\lambda)x_2, u) &\leq \lambda f_i(x_1,u) + (1-\lambda) f_i(x_2,u) \\ \mathbf{E} f_i(\lambda x_1 + (1-\lambda)x_2, u) &\leq \mathbf{E}(\lambda f_i(x_1,u) + (1-\lambda) f_i(x_2,u)) \\ \mathbf{E} f_i(\lambda x_1 + (1-\lambda)x_2, u) &\leq \lambda \mathbf{E} f_i(x_1,u) + (1-\lambda) \mathbf{E} f_i(x_2,u) \end{aligned}$$

So $\mathbf{E} f_i$ are convex functions on x. Thus the problem is a convex optimization problem.

$L(x,y) = \mathbf{E} f_0(x,u) + \sum_{i=1}^m y_i \mathbf{E} f_i(x,u) - \delta_{\mathbf{R}^m_+}(y)$

$$\begin{aligned} g(y) &= \inf_x L(x,y) \\ &= \inf_x (\mathbf{E} f_0(x,u) + \textstyle\sum_{i=1}^m y_i \mathbf{E} f_i(x,u)) - \delta_{\mathbf{R}^m_+}(y) \\ &= \inf_x \mathbf{E}(f_0(x,u) + \textstyle\sum_{i=1}^m y_i f_i(x,u)) - \delta_{\mathbf{R}^m_+}(y) \end{aligned}$$

So the dual problem is:

$$\max_{y \succeq 0} \quad \inf_x \mathbf{E}(f_0(x,u) + \sum_{i=1}^m y_i f_i(x,u))$$

Remark: $\inf_x \mathbf{E}(F(x,u)) = \mathbf{E}(\inf_x F(x,u))$ does not always hold. One simple counter example is: suppose we have $u \in \{u_1,\dots,u_N\}, N > 1$, with $prob(u = u_i) = p_i$. Let $F(x,u) = (x-u)^2$ where $x, u \in \mathbf{R}$, then

$$\inf_x \mathbf{E}(F(x,u)) = \inf_x \sum_{i=1}^N (x-u_i)^2 p_i > 0, \text{optimal sol } \bar{x} = \mathbf{E}(u)$$

$$\mathbf{E}(\inf_x F(x,u)) = \sum_{i=1}^N \inf_x (x-u_i)^2 p_i = 0$$

$$\inf_x \mathbf{E}(F(x,u)) > \mathbf{E}(\inf_x F(x,u))$$

Therefore, generally we could not interchange the infimum and expectation in the above dual problem.

(b)For the worst-case optimization,

$$\begin{array}{ll} \min_x & \sup_{u\in U} f_0(x,u) \\ \text{s.t.} & \sup_{u\in U} f_i(x,u) \le 0, \quad i=1,\dots,m \end{array}$$

If f_i are all convex in x for each u, we know $\sup_{u\in U} f_i(x,u)$ are convex functions in x. Therefore the problem is a convex optimization problem.

$L(x,y) = \sup_{u\in U} f_0(x,u) + \sum_{i=1}^m y_i \sup_{u\in U} f_i(x,u) - \delta_{\mathbf{R}_+^m}(y)$

$$\begin{aligned} g(y) &= \inf_x L(x,y) \\ &= \inf_x(\sup_{u\in U} f_0(x,u) + \textstyle\sum_{i=1}^m y_i \sup_{u\in U} f_i(x,u)) - \delta_{\mathbf{R}_+^m}(y) \\ &= \inf_x(\sup_{u^{(i)}\in U, i=0,\dots,m}(f_0(x,u^{(0)}) + \textstyle\sum_{i=1}^m y_i f_i(x,u^{(i)}))) - \delta_{\mathbf{R}_+^m}(y) \end{aligned}$$

Generally, the infimum and supremum in the above expression can not be interchanged except that we add more assumptions. We know most of minimax theorems:

$$\inf_{x\in D}\sup_{t\in T} F(x,t) = \sup_{t\in T}\inf_{x\in D} F(x,t)$$

require the quasiconvex-quasiconcave of function F(x,t), and the compactness, convexity or connectedness of sets D and(or) T. In the above $g(y)$, $D = \bigcap_{i=0}^m \mathrm{dom} f_i, T = U \times U \times \cdots \times U, m+1$ product space of U. We already know $f_i(x,u)$ are convex in x for each u. So for example, assume that $f_i(x,u), i=0,\dots,m$ are also quasiconcave in u for each x, and continuous finite on $D\times U$, D and U are closed nonempty convex sets(in fact we already know D is convex), either D or U is bounded. Then the above infimum and supremum can interchange. Then the dual problem is:

$$\max_{y\succeq 0} \quad \sup_{u_i\in U,\ i=0,\dots,m} (\inf_x (f_0(x,u_0) + \sum_{i=1}^m y_i f_i(x,u_i)))$$

In fact, if we are futherly given the explicit form of $f_i, i=1,\dots,m$, we can simplify the above dual problem. For example, if $f_i, i=1,\dots,m$ are affine functions, the dual will just maximize the supremum of the conjugate function of f_0.

(c)When u has a finite number of possible values,i.e., $u \in \{u_1, \ldots, u_N\}$, with $prob(u = u_i) = p_i$, we can rewrite the stochastic optimization problem and worst-case optimization problem explicitly as follows.
Stochastic optimization:

$$\min_{x \in \mathbf{R}^n} \quad \sum_{j=1}^{N} f_0(x, u_j) p_j$$

$$\text{s.t.} \quad \sum_{j=1}^{N} f_i(x, u_j) p_j \leq 0, \quad i = 1, \ldots, m$$

Worst-case optimization:

$$\begin{aligned} &\min \quad t \\ &\text{s.t.} \quad f_0(x, u_j) \leq t, \quad j = 1, \ldots, N \\ &f_i(x, u_j) \leq 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, N \end{aligned}$$

□