Optimal Consumption
Problem 4.58 from Boyd

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MATH 583 - Convex Optimization

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Problem: In this problem we consider the optimal way to consume (or spend) an initial amount of money (or other asset) k_0 over time. The variables are $c_1, ..., c_T$, where $c_t \geq 0$ denotes the consumption in period t. The utility derived from a consumption level c is given by u(c), where u : R → R is an increasing concave function. The present value of the utility derived from the consumption is given by

$$U = \sum_{t=1}^{T} \beta^t u(c_t)$$

where $0 < \beta < 1$ is a discount factor.

Let k_t denote the amount of money available for investment in period t. We assume that it earns an investment return given by f(k_t), where f : R → R is an increasing, concave investment return function, which satisfies $f(0) = 0$. For example if the funds earn simple interest at rate R percent per period, we have $f(a) = (R/100)a$. The amount to be consumed, i.e., c_t, is withdrawn at the end of the period, so we have the recursion

$$k_{t+1} = k_t + f(k_t) - c_t; \quad t = 0, ..., T$$

The initial sum $k_0 > 0$ is given. We require $k_t \geq 0, t = 1, ..., T + 1$ (but more sophisticated models, which allow $k_t < 0$, can be considered). Show how to formulate the problem of maximizing U as a convex optimization problem. Explain how the problem you formulate is equivalent to this one, and exactly how the two are related.[1]

Solution: So, our maximization problem can be written as:

$$\begin{aligned} &\text{Minimize } -\textstyle\sum_{t=1}^{T} \beta^t u(c_t) \\ &\text{subject to } k_{t+1} = k_t + f(k_t) - c_t; \quad t = 0, ..., T \\ &c_t \geq 0; \quad t = 0, ..., T \\ &k_t \geq 0; \quad t = 0, ..., T + 1 \end{aligned}$$

Now, we know that if $\exists z \epsilon R^m$, and $\varphi : R^m \to R^T$ which is one-to-one and onto our problem's domain (here, it is R^n_+), then an equivalent problem is:

$$\begin{gathered}\text{Minimize } -\textstyle\sum_{t=1}^{T} \beta^t u(\varphi_t(z)) \\ \text{subject to } k_{t+1} = k_t + f(k_t) - \varphi_t(z); t = 0, ..., T \\ \varphi_t(z) \geq 0; t = 0, ..., T \\ k_t \geq 0; t = 0, ..., T+1\end{gathered}$$

where $\varphi(z) = [\varphi_1(z), \varphi_2(z), ..., \varphi_T(z)]^T$. But notice that our first constraint tells use that we can write

$$\varphi(z) = Ak + F(k)$$

where A is a $(T+1) \times T$ matrix where $a_{i,i} = 1, a_{i,i+1} = -1$, and zero everywhere else, and F is the vector valued function $F(k) = [f(k_0), f(k_1), ...f(k_T)]^T$.

Clearly φ is onto, but in general will not necessarily be one-to-one. However, we can truncate the matrix to make it one-to-one in this case. Notice that our objective function is minimizing only over consumption in periods 1 through T, and not in period T+1. Therefore, we know that our optimal capital vector k must have $k_{T+1} = 0$, as it is always better to consume everything you have in the last period. Thus, we can truncate the matrix A by deleting the last column. Now, the new matrix $\hat{A}$ is bi-diagonal, and so invertible. This makes φ one-to-one and we can write our optimization problem as

$$\begin{gathered}\text{Minimize } -\textstyle\sum_{t=1}^{T} \beta^t u((k_t - k_{t+1}) + f(k_t)) \\ \text{subject to } -k_{t+1} + k_t + f(k_t) \geq 0; t = 1, ..., T \\ k_t \geq 0; t = 1, ..., T+1\end{gathered}$$

Notice that we have lost a condition (the condition $k_1 = k_0 + f(k_0) - c_0$). However, as k_0 is given, and c_0 is a free variable (we are not optimizing over it), this is a deterministic equation, and so is satisfied arbitrarily. So, as $k_t - k_{t+1}$ is linear and $f(k_t)$ is concave, then $(k_t - k_{t+1}) + f(k_t)$ is concave in k, and so $-(k_t - k_{t+1}) - f(k_t)$ is convex in k. Also, by our composition rules, as u is increasing and concave, and $(k_t - k_{t+1}) + f(k_t)$ is also concave, then $-\beta^t u((k_t - k_{t+1}) + f(k_t))$ is convex in k. So, we have that

$$\begin{gathered}\text{Minimize } -\textstyle\sum_{t=1}^{T} \beta^t u((k_t - k_{t+1}) + f(k_t)) \\ \text{subject to } k_{t+1} - k_t - f(k_t) \leq 0; t = 1, ..., T \\ -k_t \leq 0; t = 1, ..., T+1\end{gathered}$$

is a convex optimization problem. In short, we were able to write this as a convex optimization problem due to the fact that we can relax the constraints $k_{t+1} = k_t + f(k_t) - c_t$ to $k_{t+1} \leq k_t + f(k_t) - c_t$.

Dual Problem: Now we will write the Lagrangian dual of this problem. So our Lagrangian is

$$L(c,k,\lambda,\gamma,\omega) = -\sum_{t=1}^{T} \beta^t u(c_t) - \sum_{t=0}^{T} \lambda_{t+1}(k_t - k_{t+1} + f(k_t) - c_t)$$
$$- \sum_{t=1}^{T+1} \gamma_t k_t - \sum_{t=1}^{T} \omega_t c_t + \delta_{R_+^{T+1}}(\lambda,\gamma) + \delta_{R_+^T}(\omega)$$

So, we can write the dual function as the sum of two infimums

$$g(\lambda,\gamma,\omega) = inf_c[-(\sum_{t=1}^{T} \beta^t u(c_t) - \lambda_{t+1}c_t + \omega_t c_t) + \lambda_1 c_0]$$
$$+ inf_k[-\sum_{t=0}^{T} \lambda_{t+1}(k_t - k_{t+1} + f(k_t)) - \sum_{t=1}^{T+1} \gamma_t k_t]$$
$$+ \delta_{R_+^{T+1}}(\lambda,\gamma) + \delta_{R_+^T}(\omega)$$

We can turn the infimums into supremums, and take the summations out to get

$$= -\sum_{t=1}^{T} sup_{c_t}[(\omega_t - \lambda_t)c_t + \beta^t u(c_t)] - \sum_{t=1}^{T} sup_{k_t}[(\gamma_t + \lambda_{t+1} - \lambda_t)k_t + \lambda_{t+1} f(k_t)]$$
$$- sup_{k_{T+1}}[(\gamma_{T+1} - \lambda_{T+1})k_{T+1}] - \lambda(k_0 + f(k_0) - c_0) + \delta_{R_+^{T+1}}(\lambda,\gamma) + \delta_{R_+^T}(\omega)$$

Here, we notice that the supremums can be represented as conjugates. Thus, our dual problem is

$$\text{Maximize } \sum_{t=1}^{T}[\beta^t u^*((\omega_t - \lambda_t)/\beta^t) - \lambda_{t+1} f^*((\gamma_t + \lambda_{t+1} - \lambda_t)/\lambda_{t+1})] - \lambda_1[k_0 + f(k_0) - c_0]$$

subject to $\lambda, \gamma, \omega \succeq 0$ and $(\gamma_{T+1} - \lambda_{T+1}) \neq 0$. Here, the condition $(\gamma_{T+1} - \lambda_{T+1}) \neq 0$ is known, in economics, as the transversality condition, or the optimal criteria condition for the terminal values.

References

[1] Boyd, Stephen and Lieven Vandenberghe, *Convex Optimization.* Cambridge: Cambridge UP 1994.