Problem 4.47

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Max Determinant Positive Semi Definite Matrix Completion

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Marzeih Nabi-Abdolyousefi

We consider a matrix $A \in S^2$, with some entries specified, and the others not specified. The positive semidefinite matrix completion problem is to determine values of the unspecified entries of the matrix so that $A \succeq 0$ (or to determine that such a completion does not exist).

(a) Explain why we can assume without loss of generality that the diagonal entries of A are specified.

For existence of PD completion w.l.g assume that all diagonal entries are specified, if not, we can choose arbitrary large and positive values and PD completion will relax to maximal partial principal submatrix of all whose diagonal entries are specified. For maximum determinant is obviously necessary condition because determinat of a PD matrix is monotone increasing in its diagonal entries.

(b) Show how to formulate the positive semidefinite completion problem as an SDP feasibility problem.

Set the feasibility problem:

We can introduce $A_j,\ b_j,\ \ j$: all specified entries and

$$\begin{aligned} minimize \quad s \\ -X \preceq sI \\ A_j.X = b_j \ j : 1...m \end{aligned} \tag{1}$$

If $s > 0$ the PS matrix completion problem is infeasible and if $s \leq 0$ the problem is feasible.

(c) Assume that A has at least one completion that is positive definite, and the diagonal entries of A are specified (i.e., fixed). The positive definite completion with largest determinant is called the maximum determinant completion. Show that the maximum determinant completion is unique. Show that if $A^{\star}$ is the maximum determinant completion, then $(A^{\star})^{-}1$ has zeros in all the entries of the original matrix that were not specified. Hint. The gradient of the function $f(X) = log|X|$ is $\bigtriangledown f(X) = X^{(-1)}$

The Optimization problem is:

$$\begin{aligned} maximize \;\; & |X| \\ X &\succeq 0 \\ A_j.X &= b_j \end{aligned} \tag{2}$$

S: set of all positive semidefinite completions of X. We know that S is:
1. nonempty
2. closed
3. bounded (off-diagonal entries cannot be too large because diagonal entries are specified and any 2-2 principal submatrix must be positive semidefinite.)
4. CVX, Any solution is in the relative interior of S $ri(S)$ because the boundary consists of PSD matrices and their determinant is zero. $|X|$ is a log concave function over a PD matrices, so this optimization problem has a unique solution at a unique point in the set S.

Now if we differentiate the determinant with respect to an unspecified entry x_{ij}, we will have:

$$\frac{\partial \; |X|}{\partial \; x_{ij}} = 2(-1)^{i+j} \|X(i/j)\| \tag{3}$$

$X(i/j)$ is submatrix of X resulting from deleting row i and column j. x_{ij} is unspecified, so $\frac{\partial \; |X|}{\partial \; x_{ij}} = 0$. Then we will have $|X(i/j)| = 0$ and it means ij^{th} and ji^{th} elements in the X^{-1} are zero (related co-factor elements are zero.)

We can represent the cvx optimization problem in this form:

$$\begin{aligned} minimize \;\; & -log|X| \\ X &\succeq 0 \\ A_j.X &= b_j \end{aligned} \tag{4}$$

Also we can use the optimization problem set (4), so we will have $\nabla \; f =$

$log|X|$:

$$\nabla\ f = X^{-1}$$
$$\frac{\partial\ f(X)}{\partial\ x_{ij}} = \frac{|X(i/j)|}{|X|} = 0 \qquad (5)$$

From equation 5 we have $|X(i/j)| = 0$.
As a result if X is a diagonal matrix,then $X^\star = X$ because form the mentioned theorem we know $A^\star$ is diagonal so $A^\star$ has to be diagonal.

The Lagrangian and dual problem are (CVX optimal set (4):

$$\begin{aligned} L(X, \lambda, \nu) &= -log|X| - \lambda.X + \sum_{i=1}^{m} \nu_i(A_i.X - b_i) \\ L(X, \lambda, \nu) &= -log|X| - \lambda.X - \nu^T b + \sum_{i=1}^{m} \nu_i A_i.X \end{aligned} \qquad (6)$$

while $\lambda \in R_+^{n\times n}$

If we differentiate the Lagrangian respect to X we will have:

$$\begin{aligned} \nabla L(X, \lambda, nu)_{\lambda\in R_+^{n\times n}} &= -X^{-1} - \lambda + \sum_{i=1}^{m} \nu_i A_i = 0 \\ X^{-1} &= \sum_{i=1}^{m} \nu_i A_i - \lambda \end{aligned} \qquad (7)$$

so we will have

$$\begin{aligned} g(\lambda, \nu)_{\lambda\in R_+^{n\times n}} &= inf_X \quad L(X, \lambda, \nu) \\ &= -log|Z^{-1}| - \lambda.Z^{-1} - \nu^T b + \sum_{i=1}^{m} \nu_i A_i.Z^{-1} \end{aligned} \qquad (8)$$

and the dual problem is:

$$\begin{array}{lll} maximize & g(\lambda, \nu, Z)_{\lambda\in R_+^{n\times n}} & \\ Z & = & \sum_{i=1}^{m} \nu_i A_i - \lambda \\ Z & \succeq 0 & \\ \lambda & \succeq 0 & \end{array} \qquad (9)$$

(d) Suppose A is specified on its tridiagonal part, i.e., we are given

$A_{11}, ..., A_{nn}$ **and** $A_{12}, ...A_{(n-1)n}$**. Show that if there exists a positive definite completion of A, then there is a positive definite completion whose inverse is tridiagonal.**

If there exists PD completion for A, then there exists unique PD completion for A that has the the maximum determinant ($A^{\star}$). From part (c) we know $A^{\star}$ has zeros in all unspecified entries of the origin matrix, so $(A^{\star})^{-1}$ is tridiagonal.