582: PROBLEM 4.16

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Minimum Fuel Optimal Control. We consider a linear dynamical system with state $x(t) \in \mathbb{R}^n, t = 0, \ldots, N$, and actuator or input signal $u(t) \in \mathbb{R}$, for $t = 0, \ldots, N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), t = 0, \ldots, N-1$$

where $A\mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are given. We assume that the initial state is zero, i.e, $x(0) = 0$.

The minimum fuel optimal control problem is to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t))$$

subject to the constraint that $x(N) = x_{\text{des}}$, where N is the given time horizon, and $x_{\text{des}} \in \mathbb{R}^n$ is the given desired target state. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the fuel use map for the actuator, and gives the amount of fuel as a function of the actuator signal amplitude. In this problem, we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1 \end{cases}$$

Formulate the minimum fuel optimal control problem as an LP.

Solution:

We have specified $x(0) = 0$ and $x(N) = x_{\text{des}}$. Combining these conditions with the specified system dynamics, we get

$$\begin{aligned} x(1) &= bu(0) \\ x(2) &= Abu(0) + bu(1) \\ x(3) &= A^2bu(0) + Abu(1) + bu(2) \\ &\vdots \\ x(N) &= A^{N-1}bu(0) + A^{N-2}bu(1) + \cdots + Abu(N-1) + bu(N-1) = x_{\text{des}} \end{aligned}$$

We now define the controllability matrix:

$$C = \begin{bmatrix} A^{n-1}b & A^{N-2}b & \cdots & Ab & b \end{bmatrix}$$

If we let $u^T = \begin{bmatrix} u(0) & \dots u(N-1) \end{bmatrix}$, we can write our initial and target conditions for the problem as follows:

$$Cu = x_{\text{des}}$$

We now turn our attention back to $f(a)$. We can add new variables and inequalities in order to turn f into a linear objective. There are several ways to do this.

First, we may add a single variable t, together with the constraints

$$\begin{aligned} |a| &\leq t \\ 2|a| - 1 &\leq t \end{aligned}$$

To see that this works, note only that for $|a| \leq 1$, we have $|a| \geq 2|a|-1$, while for $|a| > 1$, $|a| \leq 2|a| - 1$. Then for $|a| \leq 1$, the second condition is redundant, while for $|a| > 1$, the first is redundant, which gives us exactly what we want.

Rewriting above, we have the inequalities

$$\begin{aligned} -t \leq a &\leq t \\ -\frac{t+1}{2} \leq a &\leq \frac{t+1}{2} \end{aligned}$$

Following this methodology, we have added one new variable and four new inequality constraints for each $u(i)$.

We can now introduce a vector $t = [t_0, \ldots, t_{N-1}]$, one t_i for each $u(i)$. Our linear program is then

$$\mathcal{P} = \begin{cases} \text{minimize} & \underline{1}^T t \\ \text{subject to} & -t \leq u \leq t \\ & -\frac{t+\underline{1}}{2} \leq u \leq \frac{t+\underline{1}}{2} \\ & Cu = x_{\text{des}} \end{cases}$$

We form the Lagrangian

$$\begin{aligned} L(t, u, \alpha, \beta, \gamma, \Delta, \theta) &= \underline{1}^T t - \alpha^T(u+t) + \beta^T(u-t) - \gamma^T(2u+1+t) \\ &\quad + \Delta^T(2u-1-t) + \theta^T(Cu - x_{\text{des}}) + \delta_{\mathbb{R}_+^{4N}}(\begin{bmatrix}\alpha \\ \beta \\ \gamma \\ \Delta\end{bmatrix}) \\ &= (-\alpha + \beta - 2\gamma + 2\Delta + C^T\theta)^T u + (\underline{1} - \alpha - \beta - \gamma - \Delta)^T t \\ &\quad - (\gamma^T \underline{1} + \Delta^T \underline{1} + \theta^T x_{\text{des}}) + \delta_{\mathbb{R}_+^{4N}}(\begin{bmatrix}\alpha \\ \beta \\ \gamma \\ \Delta\end{bmatrix}) \end{aligned}$$

Now we find the dual objective function:

$$\begin{aligned} g(\alpha, \beta, \gamma, \Delta, \theta) &= \inf_{u,t} L = \inf_u (-\alpha + \beta - 2\gamma + 2\Delta + C^T\theta)^T u \\ &\quad + \inf_t [\underline{1} - (\alpha + \beta + \gamma + \Delta)]^T t - (\gamma^T \underline{1} + \Delta^T \underline{1} + \theta^T x_{\text{des}}) + \delta_{\mathbb{R}_+^{4N}}(\begin{bmatrix}\alpha \\ \beta \\ \gamma \\ \Delta\end{bmatrix}) \end{aligned}$$

Looking at the t part of the expression, it is clear that we must have $\alpha + \beta + \gamma + \Delta \leq \underline{1}$ in order for g to be finite, since $t \geq 0$. With this condition, the infimum over t is zero.

By similar logic, looking at the u part gives us $-\alpha+\beta-2\gamma+2\Delta+C^T\theta = 0$, in order to ensure that g is finite. In this case the infimum again is 0. Then under the conditions just derived, we have

$$g(\alpha,\beta,\gamma,\Delta,\theta) = -(\gamma^T\underline{1} + \Delta^T\underline{1} + \theta^T x_{\text{des}}) + \delta_{\mathbb{R}_+^{4N}}\left(\begin{bmatrix}\alpha\\ \beta\\ \gamma\\ \Delta\end{bmatrix}\right)$$

Then our dual problem is

$$\mathcal{D} = \begin{cases} \text{maximize} & -\gamma^T\underline{1} - \Delta^T\underline{1} - \theta^T x_{\text{des}} \\ \text{subject to} & 0 \leq \alpha,\beta,\gamma,\Delta \\ & C^T\theta = \alpha - \beta + 2\gamma - 2\Delta \\ & \alpha + \beta + \gamma + \Delta \leq \underline{1} \end{cases}$$