Portfolio Modeling Using LPs

LP Modeling Technique

Step 1: Determine the decision variables and label them. The decision variables are those variables whose values must be determined in order to execute a plan of action or strategy regardless of the optimality of this plan.

Step 2: Determine the objective and write an expression for it that is linear in the decision variables.

Step 3: Determine the explicit (or stated) constraints and write them as either equality or inequality constraints that are linear in the decision variables.

Step 4: Determine the implicit constraints. These are the constraints that are not explicitly stated but are required in order for the problem to be physically meaningful. These constraints must also be written as either equality or inequality constraints that are linear in the decision variables.

Toy Problem

You have \$20,000 to invest in four different financial instruments.

- Buy stock X which is currently selling for \$20 per share.
- Purchase *European call* options to buy a share of stock X at \$15 in exactly 6 months time. These options are selling today for \$10 each.
- Raise more funds for investment immediately by selling the European call options described above under the same terms.
- Purchase 6 month riskless zero-coupon bonds having a face value of \$100 at a cost of \$90 each today.

Remark (European Call Options)

A European call option to buy a stock at a price x and *exercise date* d gives the purchaser the right to buy the stock at x dollars on precisely the date d. The purchaser of the option need not exercise this right to buy, and they can only exercise this right on the date d (not before or after).

Scenarios for the sale price of stock X: equally likely

Scenario 1: Stock X sells for $20 a share in 6 months.

Scenario 2: Stock X sells for $40 a share in 6 months.

Scenario 2: Stock X sells for $12 a share in 6 months.

Option Restrictions: Due to the heavy risks involved with selling options, the exchange has placed a limit on the total number of European calls on stock X that you can sell of 5000.

Formulate an LP to determine the portfolio of stocks, bonds, and options that maximizes your expected profit in 6 months.

Determine the Decision Variables.

$$
\begin{aligned}
S &= \text{the number of stock X purchased} \\
B &= \text{the number of bonds purchased} \\
C_1 &= \text{the number of call options purchased} \\
C_2 &= \text{the number of call options sold}
\end{aligned}
$$

Determine the Objective

Maximize Expected Profit

Bonds: \$10 per bond

Stock:

Scenario 1: the profit is \$0.

Scenario 2: the profit is \$20.

Scenario 3: the profit is -\$8.

$$\text{Expected Profit} = \frac{1}{3}(0) + \frac{1}{3}(20) + \frac{1}{3}(-8) = 4.$$

Call Options Bought:

Scenario 1: Exercise option, buy the stock for \$15, then sell it again for \$20: profit=\$5. Subtract the cost of the stock (\$10): overall profit = -\$5.

Scenario 2: Exercise option, buy the stock for \$15, then sell it again for \$40: profit = \$25. Subtract cost of the stock (\$10): overall profit = \$15.

Scenario 3: Do not exercise option: overall profit = -\$10.

$$\text{Expected Profit} = \frac{1}{3}(-5) + \frac{1}{3}(15) + \frac{1}{3}(-10) = 0.$$

Call Options Sold:

Scenario 3: Purchaser of buy option exercises this option. We must buy the stock for \$20 and then turn around and sell it to the purchaser of the option for \$15, thus incurring a loss of \$5. However, the purchaser initially paid us \$10 for the option: overall profit = \$5.

Scenario 3: Purchaser of the option to buy again exercises this option. In this case, we must buy the stock for \$40 and then turn around and sell it to the purchaser of the option for \$15, thus incurring a loss of \$25. However, the purchaser initially paid us \$10 for the option: overall profit = -\$15.

Scenario 3: Purchaser of the option does not exercise the option and so our profit in this scenario is the \$10 the purchaser initially gave us for the option: overall profit =\$10

$$\text{Expected Profit} = \frac{1}{3}(5) + \frac{1}{3}(-15) + \frac{1}{3}(10) = 0.$$

Objective: Maximize $10B + 4S$.

Explicit Constraints

Budget Constraint: $90B + 20S + 10C_1 \ \leq \ 20000 + 10C_2$

Margin Constraint: $C_2 \ \leq \ 5000$

Implicit Constraints

$$0 \leq S, \ 0 \leq B, \ 0 \leq C_1, \text{ and } 0 \leq C_2.$$

LP Model

$$\begin{array}{ll} \text{maximize} & 10B \ + \ 4S \\ \text{subject to} & 90B + 20S + 10(C_1 - C_2) \ \leq \ 20000 \\ & 0 \leq S, \ 0 \leq B, \ 0 \leq C_1, \text{ and } 0 \leq C_2 \leq 5000. \end{array}$$

Simplification

Observe that if $C_2 < 5000$ at the solution and N is any number between C_2 and 5000, then we may change C_2 to $C_2 + N$ and C_1 to $C_1 + N$ and obtain a new solution that must also be optimal.

The effect here is to buy N calls and turn around and immediately sell them again. This is of course wasteful activity that is not prohibited in our modeling.

To avoid this we introduce a new variable $C = C_1 - C_2$. Negative values of C correspond to selling calls, while positive values of C corresponds to buying them.

New LP

$$\begin{array}{ll} \text{maximize} & 10B \ + \ 4S \\ \text{subject to} & 90B + 20S + 10C \ \leq \ 20000 \\ & 0 \leq S, \ 0 \leq B, \ -5000 \leq C. \end{array}$$

Solution

$$\begin{aligned} B &= 0 \\ S &= 3500 \\ C &= -5000, \end{aligned}$$

with optimal value giving an expected profit of \$14000.

But remember that this is only the *expected* profit. In the event that scenario 2 actually occurs, we actually incur a loss of \$5000 since

$$20S + 15C = 20 \times 3500 - 15 \times 5000 = -5000.$$

An Alternative

Add the condition that the profit from the portfolio must be at least \$2000 by adding further constraints that require that regardless of the scenario the profit must exceed \$2000. To do this, we simply lower bound the revenue at the end of 6 month by \$22000.

$$\text{Scenario 1: } 100B + 20S + 5C \geq 22000$$

$$\text{Scenario 2: } 100B + 40S + 25C \geq 22000$$

$$\text{Scenario 3: } 100B + 12S \geq 22000.$$

New LP

$$\begin{array}{lr} \text{maximize} & 10B \ + \ 4S \\ \text{subject to} & 90B + 20S + 10C \leq 20000 \\ & 100B + 20S + 5C \geq 22000 \\ & 100B + 40S + 25C \geq 22000 \\ & 100B + 12S \geq 22000 \\ & 0 \leq S, \ 0 \leq B, \ -5000 \leq C. \end{array}$$

Solution

$$\begin{aligned} B &= 0 \\ S &= 2800 \\ C &= -3600, \end{aligned}$$

with optimal value giving an expected profit of \$11200.

But again it must be remembered that this is only the *expected* profit. Indeed, if scenario 2 occurs, then the actual profit will be \$2000 which is the least amount required.

Another Alternative

To overcome the potential downside of any given scenario, we may choose instead to maximize the smallest possible profit regardless of which scenario occurs. We can do this by introducing a new variable Z representing the least possible revenue that we may have at the end of 6 months, regardless of which scenario occurs. We then maximize Z.

New LP

$$\begin{array}{lr} \text{maximize} & Z \\ \text{subject to} & 90B + 20S + 10C \leq 20000 \\ & 100B + 20S + 5C \geq 20000 + Z \\ & 100B + 40S + 25C \geq 20000 + Z \\ & 100B + 12S \geq 20000 + Z \\ & 0 \leq S,\ 0 \leq B,\ -5000 \leq C. \end{array}$$

Solution

$$\begin{aligned} B &= 0 \\ S &= 2273 \\ C &= -2545, \end{aligned}$$

giving an expected profit of \$9092. The least possible profit from this solution is \$7285.