Engineering Design Example

Optimal proportions of a can.

A cylindrical can of a given volume V_0 is to be proportioned in such a way as to minimize the total cost of the material in a box of 12 cans, arranged in a 3×4 pattern.

The cost expression takes the form

$$c_1 S_1 + c_2 S_2,$$

where S_1 is the surface area of the 12 cans and S_2 is the surface area of the box. (The coefficients c_1 and c_2 are positive.)

A side requirement is that no dimension of the box can exceed a given amount D_0.

design parameters:

$$r = \text{radius of can},\ h = \text{height of can}$$

volume constraint:

$$\pi r^2 h = V_0 \quad (\text{or} \quad \pi r^2 h \geq V_0, \text{ see below!})$$

surface area of cans:

$$S_1 = 12(2\pi r^2 + 2\pi r h) = 24\pi r(r + h)$$

box dimensions:

$$8r \times 6r \times h$$

surface area of box:

$$S_2 = 2(48r^2 + 8rh + 6rh) = 4r(24r + 7h)$$

size constraints:

$$8r \leq D_0, \quad 6r \leq D_0, \quad h \leq D_0$$

nonnegativity constraints:

$$r \geq 0, \quad h \geq 0 \quad (!)$$

*Summary.*The design choices that are available can be identified with the set C consisting of all the pairs $(r, h) \in I\!R^2$ that satisfy the conditions

$$r \geq 0, \quad h \geq 0, \quad 8r \leq D_0, \quad 6r \leq D_0, \quad h \leq D_0, \quad \pi r^2 h = V_0.$$

Over this set we wish to minimize the function

$$f_0(r, h) = c_1[24\pi r(r + h)] + c_2[4r(24r + 7h)] = d_1 r^2 + d_2 r h,$$

where $d_1 = 24\pi c_1 + 96 c_2$ and $d_2 = 24\pi c_1 + 28 c_2$.

Comments

Redundant constraints:

$$8r \leq D_0, \quad 6r \leq D_0, \quad h \leq D_0$$

It is obvious that the condition $6r \leq D_0$ is implied by the condition $8r \leq D_0$ and therefore could be dropped without affecting the problem. But in problems with many variables and constraints such redundancy may be hard to recognize. From a practical point of view, the elimination of redundant constraints could pose a challenge as serious as that of solving the optimization problem itself.

Inactive constraints:

It could well be true that the optimal pair (r, h) (unique??) is such that either the condition $8r \leq D_0$ or the condition $h \leq D_0$ is satisfied as a strict inequality, or both. In that case the constraints in question are inactive in the local characterization of optimal point, although they do affect the shape of the set C. Again, however, there is little hope, in a problem with many variables and constraints, of determining by some preliminary procedure just which constraints will be active and which will not. This is the crux of the difficulty in many numerical approaches.

Redundant variables:

It is be possible to solve the equation $\pi r^2 h = V_0$ for h in terms of r and thereby reduce the given problem to one in terms of just r, rather than (r, h). However, besides being a technique that is usable only in special circumstances, the elimination of variables from (generally nonlinear) systems of equations is not necessarily helpful. There may be a trade-off between the lower dimensionality achieved in this way and other properties such as convexity.

Inequalities versus equations:

$$\pi r^2 h = V_0 \qquad \text{or} \qquad \pi r^2 h \geq V_0$$

The latter constraint can be used because of the nature of the cost function. While it may seem instinctive to prefer the equation to the inequality in the formulation, the inequality turns to be superior in the present case because the set C' happens to be "convex," whereas C isn't.

Convexity:
This problem is not fully of "convex" type in itself, despite the preceding remark. Nonetheless, it can be made convex by a certain change of variables. Set

$$w = rh.$$

Then the optimization problem becomes

$$\begin{array}{ll} \text{minimize} & d_1 r^2 + d_2 w \\ \text{subject to} & rw \geq V_0/\pi \\ & 8r \leq D_0,\ w \leq D_0 r \\ & 0 \leq r,\ 0 \leq w \end{array}$$