Linear Difference Equations

Constant Coefficient LDEs

Let x_i be a (scalar) sequence defined for $i \geq 0$.

A k-step linear difference equation is an equation of the form

$$\text{(LDE)} \qquad x_{i+k} + \alpha_{k-1} x_{i+k-1} + \cdots + \alpha_0 x_i = b_i \qquad (i \geq 0).$$

If $b_i \equiv 0$, the linear difference equation (LDE) is said to be homogeneous, in which case we will refer to it as (lh).

If $b_i \neq 0$ for some $i \geq 0$, the linear difference equation (LDE) is said to be inhomogeneous , in which case we refer to it as (li).

Initial Value Problem (IVP)

Given x_i for $i = 0, \ldots, k-1$, determine x_i satisfying (LDE) for $i \geq 0$.

Theorem: *There exists a unique solution of (IVP) for (lh) or (li).*

Theorem: *The solution set of (lh) is a k-dimensional vector space (a subspace of the set of all sequences $\{x_i\}_{i \geq 0}$).*

Characteristic Polynomial for LDEs

Define the *characteristic polynomial* of (*lh*) to be

$$p(r) = r^k + \alpha_{k-1} r^{k-1} + \cdots + \alpha_0.$$

Assume that $\alpha_0 \neq 0$. (If $\alpha_0 = 0$, (LDE) isn't really a k-step difference equation since we can shift indices and treat it as a $\widetilde{k}$-step difference equation for a $\widetilde{k} < k$, namely $\widetilde{k} = k - \nu$, where ν is the smallest index with $\alpha_\nu \neq 0$.)

Let $r_1, \ldots, r_s$ be the distinct zeroes of p, with multiplicities $m_1, \ldots, m_s$. Note that each $r_j \neq 0$ since $\alpha_0 \neq 0$, and $m_1 + \cdots + m_s = k$. Then a basis of solutions of (*lh*) is:

$$\left\{ \{i^l r_j^i\}_{i=0}^{\infty} \,:\, 1 \le j \le s,\ 0 \le l \le m_j - 1 \right\}.$$

Example: Fibonacci Sequence

$$F_{i+2} - F_{i+1} - F_i = 0, \quad F_0 = 0, \quad F_1 = 1.$$

The associated characteristic polynomial $r^2 - r - 1 = 0$ has roots $r_\pm = \frac{1 \pm \sqrt{5}}{2}$ ($r_+ \approx 1.6, r_- \approx -0.6$). The general solution of (lh) is

$$F_i = C_+ \left(\frac{1+\sqrt{5}}{2} \right)^i + C_- \left(\frac{1-\sqrt{5}}{2} \right)^i.$$

The initial conditions $F_0 = 0$ and $F_1 = 1$ imply that $C_+ = \frac{1}{\sqrt{5}}$ and $C_- = -\frac{1}{\sqrt{5}}$. Hence

$$F_i = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^i - \left(\frac{1-\sqrt{5}}{2} \right)^i \right).$$

Since $|r_-| < 1$, we have $\left(\frac{1-\sqrt{5}}{2} \right)^i \to 0$ as $i \to \infty$. Hence, the sequence behaves asymptotically like $\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^i$.

Conversion to 1-Step Vector LDE

Define $\widetilde{x}_i = \begin{bmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_{i+k-1} \end{bmatrix}$. Then $\widetilde{x}_{i+1} = A\widetilde{x}_i$ for $i \geq 0$, where

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_0 & \dots & & -\alpha_{k-1} \end{bmatrix},$$

and $\widetilde{x}_0 = (x_0,\ x_1,\ \dots,\ x_{k-1})^T$ is given by the I.C. So (lh) is equivalent to the one-step vector difference equation

$$\widetilde{x}_{i+1} = A\widetilde{x}_i, \quad i \geq 0,$$

whose solution is $\widetilde{x}_i = A^i\widetilde{x}_0$.

Solutions to 1-Step Vector LDEs

The characteristic polynomial of (lh) is the characteristic polynomial of A.

Because A is a companion matrix, it is nonderogatory.

If $A = PJP^{-1}$ is the Jordan decomposition of A, then

$$\widetilde{x}_i = PJ^iP^{-1}\widetilde{x}_0,$$

or, equivalently,

$$\widetilde{y}_i = J^i\widetilde{y}_0 \quad \text{where} \quad \widetilde{y}_i = P^{-1}\widetilde{x}_i.$$

Solutions to 1-Step Vector LDEs

Let J_j be the $m_j \times m_j$ block corresponding to r_j (for $1 \leq j \leq s$), so $J_j = r_j I + N_j$, where N_j denotes the $m_j \times m_j$ shift matrix:

$$N_j = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}.$$

Then

$$J_j^i = (r_j I + N_j)^i = \sum_{\ell=0}^{i} \binom{i}{\ell} r_j^{i-\ell} N_j^{\ell}.$$

Since $\binom{i}{\ell}$ is a polynomial in i of degree ℓ and $N_j^{m_j} = 0$, we see entries in $\widetilde{y}_i$ are of the form

$$(\text{constant}) i^l r_j^i \quad \text{for} \quad 0 \leq l \leq m_j - 1,$$

which implies the same is true for the entries of $\widetilde{x}_i$ if $x_0 = P y_0$ for some y_0.

Stability of LDEs

All solutions $\{x_i\}_{i \geq 0}$ of (lh) stay bounded (i.e. are elements of l^∞)

$\Leftrightarrow$ the matrix A is power bounded (i.e., $\exists\, M$ so that $\|A^i\| \leq M$ for all $i \geq 0$)

$\Leftrightarrow$ the Jordan blocks $J_1, \ldots, J_s$ are all power bounded

$\Leftrightarrow$

$$\begin{cases} & \text{(a)} \quad \text{each } |r_j| \leq 1 & (\text{for } 1 \leq j \leq s) \\ \text{and} & \text{(b)} \quad \text{for any } j \text{ with } m_j > 1 \text{ (multiple roots)}, & |r_j| < 1 \end{cases}.$$

If (a) and (b) are satisfied, then the map $\widetilde{x}_0 \mapsto \{x_i\}_{i \geq 0}$ is a bounded linear operator from $\mathbb{R}^k$ (or $\mathbb{C}^k$) into l^∞.

Spectral Radius: $\quad \rho(A) := \max\{|\lambda| \,:\, \det(A - \lambda I) = 0\}.$