Math 555 Winter 2015

**Homework Set 8** **Due Friday, March 13**

(1) Compute the Fourier series for the following functions, and discuss the convergence of these series.

(a) $x(t) = x(t + 2\pi)$, where on the interval $[-\pi, \pi)$

$$x(t) := \begin{cases} 0 & -\pi \le t < 0, \\ \sin t & 0 \le t < \pi. \end{cases}$$

(b) $x(t) = x(t + 2\pi)$, where on the interval $[-\pi, \pi)$, $x(t) = t^2$.

(c) Show that $t^3 - \pi^2 t = 12 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin(kt)$ on $[-\pi, \pi)$. (Hint: differentiate)

(2) Let $\lambda \in \mathbb{C}$. Show that if there is a non-zero solution to $v'' = -\lambda v$ that is $2\pi$ periodic, then $\lambda = n^2$ for some $n \in \{1, 2, \dots\}$.

(3) Apply Parseval's relation to the function $f(x) = x$ on $(-\pi,\ \pi)$ to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(4) How would you define the Fourier series for the periodic function $x(t) = x(t + 10)$, where in the interval $[0, 10)$

$$x(t) := \begin{cases} 1 & 0 \le t \le 1, \\ 0 & 1 < t < 2, \\ -1 & 2 \le t \le 3, \\ 0 & 3 < t < 10 \end{cases} \qquad ?$$

Using your definition, compute the Fourier series. (Hint: Note that $x$ is a linear combination of a function and the shift of itself.)

(5) (a) Show that

$$\left\{ \sqrt{\frac{2}{\pi}} \sin(nx) \mid n = 1, 2, \dots \right\}$$

is a complete orthonormal system in $L^2(0, \pi)$.

(b) Show that if $f \in C^1[0, \pi]$ satisfies $f(0) = f(\pi) = 0$, then the expansion of $f$ interms of the orthonormal basis in a) converges uniformly to $f$ on $[0, \pi]$ (called the Fourier sine series).

(6) Let $f : \mathbb{R} \mapsto \mathbb{C}$ be integrable ($f \in L^1(-\infty, \infty)$). The *Fourier transform* (FT) of $f$ is given by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) \mathsf{e}^{-2\pi i x \xi} dx \quad \forall \xi \in \mathbb{R} .$$

(a) Show that $\mathcal{F}$ maps $L^2(-\infty, \infty)$ to $L^2(-\infty, \infty)$.

(b) Show that $\mathcal{F}$ is a unitary linear transformation on $L^2(-\infty, \infty)$ ($\|\hat{f}(\xi)\| = \|f\|$).

(c) Show that $\mathcal{F}^\star$, the adjoint of $\mathcal{F}$, is given by

$$\mathcal{F}^\star(\hat{f})(x) := f(x) := \int_{-\infty}^{\infty} \hat{f}(\xi) \mathsf{e}^{2\pi i x \xi} d\xi \quad \forall \in \mathbb{R} .$$

(d) Let $\sigma_{x_0} : L^2(-\infty, \infty) \mapsto L^2(-\infty, \infty)$ be the mapping $\sigma_{x_0}(f)(x) := f(x + x_0)$, and $\mu_{\xi_0} : L^2(-\infty, \infty) \mapsto L^2(-\infty, \infty)$ the mapping $\mu_{\xi_0}(f)(x) := \mathsf{e}^{2\pi i x \xi_0} f(x)$. Establish the following properties of $\mathcal{F}$.

(i) $\mathcal{F} \circ \sigma_{x_0} = \mu_{x_0} \circ \mathcal{F}$.

(ii) $\mathcal{F} \circ \mu_{\xi_0} = \sigma_{-\xi_0} \circ \mathcal{F}$.

(iii) $\mathcal{F}(\bar{f})(\xi) = \overline{\hat{f}(-\xi)}$

(iv) Given $f, \hat{f}, g, \hat{g} \in L^1(-\infty,\infty) \cap L^2(-\infty,\infty)$ and define the convolution of $f$ and $g$ to be

$$(f \star g)(x) := \int_{-\infty}^{\infty} f(y)\sigma_{-y}(g)(x)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy \ .$$

Show that $\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g)$.

(e) What happens to the results given above if we now define $\mathcal{F}$ on $L^1(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)\mathsf{e}^{-2\pi i\langle x,\,\xi\rangle} d\lambda_n(x) \quad \forall\, \xi \in \mathbb{R}^n \ .$$

(7) **(Bonus Problem: 10 points)** This problem analyzes the Fourier series solution to the vibrating string problem:

| | | |
|---|---|---|
| (DE) | $u_{tt} = u_{xx}$ | $0 \le x \le \pi,\ 0 \le t$ |
| (IC) | $u(x,0) = f(x), \quad u_t(x,0) = 0$ | $0 \le x \le \pi$ |
| (BC) | $u(0,t) = u(\pi,t) = 0$ | $0 \le t$ . |

(a) Show directly that if $f \in C^2[0,\pi]$, then the series for $u$ obtained by superposing fundamental modes converges uniformly on $[0,\pi] \times \mathbb{R}$ to a continuous function $u(x,t)$ that satisfies $u(x,0) = f(x)$.

(b) Show that the function $u$ from part (a) agrees with d'Alembert's solution to this IBVP, and hence show that $u \in C^2\left([0,\pi] \times \mathbb{R}\right)$, and that $u$ satisfies the wave equation.

Note: Observe that there are difficulties in trying to justify term-by-term differentiation of the series for $u$ to check that $u$ satisfies the wave equation. Find conditions of $f$ (e.g. smoothness, values of $f$, $f'$, etc. at $0$, $\pi$) which would justify this approach.)