(1) Let \( \lambda \in \mathbb{C} \). Show that if there is a non-zero solution to \( v'' = -\lambda v \) that is \( 2\pi \) periodic, then \( \lambda = n^2 \) for some \( n \in \{1, 2, \ldots \} \).

(2) Apply Parseval’s relation to the function \( f(x) = x \) on \( (-\pi, \pi) \) to evaluate \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

(3) This problem analyzes the Fourier series solution to the vibrating string problem:

\[
\begin{align*}
(\text{DE}) & \quad u_{tt} = u_{xx} & & 0 \leq x \leq \pi, \ 0 \leq t \\
(\text{IC}) & \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0 & & 0 \leq x \leq \pi \\
(\text{BC}) & \quad u(0, t) = u(\pi, t) = 0 & & 0 \leq t .
\end{align*}
\]

(a) Show that

\[
\left\{ \sqrt{\frac{2}{\pi}} \sin(nx) \mid n = 1, 2, \ldots \right\}
\]

is a complete orthonormal system in \( L^2(0, \pi) \).

(b) Show that if \( f \in C^1[0, \pi] \) satisfies \( f(0) = f(\pi) = 0 \), then the expansion of \( f \) in terms of the orthonormal basis in a) converges uniformly to \( f \) on \( [0, \pi] \) (called the Fourier sine series).

(c) Show directly that if \( f \in C^2[0, \pi] \), then the series for \( u \) obtained by superposing fundamental modes converges uniformly on \( [0, \pi] \times \mathbb{R} \) to a continuous function \( u(x, t) \) that satisfies \( u(x, 0) = f(x) \).

(d) Show that the function \( u \) from part c) agrees with d’Alembert’s solution to this IBVP, and hence show that \( u \in C^2([0, \pi] \times \mathbb{R}) \), and that \( u \) satisfies the wave equation.

Note: Observe that there are difficulties in trying to justify term-by-term differentiation of the series for \( u \) to check that \( u \) satisfies the wave equation. Find conditions of \( f \) (e.g. smoothness, values of \( f, f' \), etc. at \( 0, \pi \) which would justify this approach.)

(4) Consider a rod insulated so that no heat flows out of either end of the rod.

(a) Convince yourself that this physical situation gives rise to the following problem:

\[
\begin{align*}
(\text{DE}) & \quad u_t = u_{xx} & & 0 \leq x \leq \pi, \ 0 \leq t \\
(\text{IC}) & \quad u(x, 0) = f(x) & & 0 \leq x \leq \pi \\
(\text{BC}) & \quad u_x(0, t) = u_x(\pi, t) = 0 & & 0 \leq t .
\end{align*}
\]

(b) Separate variables to find the fundamental modes \( u(x, t) \).

(c) Show that the resulting initial states form a complete orthonormal system in \( L^2(0, \pi) \). The associates series are Fourier cosine series.

(5) Consider the Dirichlet problem for the Laplacian \( \Delta = \partial_x^2 + \partial_y^2 \) on the unit disk \( D = \{ (x, y) \mid x^2 + y^2 \leq 1 \} \), i.e., given \( f \) on \( \partial D = S^1 \), find \( u \) on \( D \) satisfying

\[
(\text{DE}) \quad \Delta u = 0, \quad (BC) \quad u|_{\partial D} = f .
\]

(a) Write \( \Delta \) in polar coordinates \((r, \theta)\) and separate variables to find solutions of the form \( u(r, \theta) = v(r)w(\theta) \).

Note: The function \( u \) should be bounded near the origin. To solve the equation for \( v \), look up Euler’s equation in an ODE book.
(b) Suppose $f \in L^2(S^1)$. Write $f$ in a Fourier series and derive a series for $u$. Show that this series converges for $r < 1$ to a $C^2$ solution of $\Delta u = 0$ satisfying the (BC) in the sense that

$$\|u(r, \cdot) - f(\cdot)\|_{L^2(S^1)} \to 0 \quad \text{as} \ r \to 1.$$