Math 555 Winter 2015

Homework Set 7 **Due Friday, March 6**

(1) Recall that $L^p(\mathbb{R}^n)$ consists of equivalence classes of functions equal a.e.
  (a) Show that any such equivalence class can contain at most one continuous function (so it makes sense to say that an element of $L^p(\mathbb{R}^n)$ is continuous).
  (b) Show that if $f \in L^\infty(\mathbb{R}^n)$ and f is continuous, then

$$\|f\|_\infty = \sup_{x\in\mathbb{R}^n} |f(x)| \ .$$

(2) Find a continuous function f on $\mathbb{R}$ so that $f \in L^1(\mathbb{R})$ but $f \notin L^\infty(\mathbb{R})$.

(3) True or False: Give a proof, or provide a counterexample.
Suppose $\{x_n\}$, $\{y_n\}$ are sequences in a Hilbert space $\mathcal{H}$, and $x, y \in \mathcal{H}$.
  (a) If $x_n \to x$ weakly and $y_n \to y$ strongly, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.
  (b) If $x_n \to x$ weakly and $y_n \to y$ weakly, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

(4) (a) Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space, let $C \subset V$ be closed and non-empty, and let $x \in V$. Show that there is a closest point in C to x. Show, however, that the closest point need not be unique even if C is convex.

(5) (a) Let $\{p_n(x)\}$ be obtained from $\{x^n : n = 0, 1, 2, \dots\}$ by Gram-Schmidt in $L^2([-1, 1])$. The polynomials

$$\left\{ \sqrt{\frac{2}{2n+1}}\, p_n \right\}$$

are called the Legendre polynomials. Show that $\{p_n\}$ is a complete orthonormal set. (Hint: Use the Weierstrass Approximation Theorem.)
  (b) Compute $p_0,\ p_1,\ p_2$.
  (c) Use (a),(b), and Hilbert space theory to find $a, b, c \in \mathbb{R}$ for which

$$\int_{-1}^{1} |x^3 - (a + bx + cx^2)|^2 dx$$

is minimized.

(6) Consider **The Projection Theorem** for nonempty closed subspaces of of a Hilbert space given on page 86 of the class notes. State and prove an analogue of this result for nonempty closed convex cones in a Hilbert space. (Hint: use your results from last term)

(7) Consider the Hilbert space $L^2(\mathbb{R}^n)$ and the set

$$K := \left\{ f \in L^2(\mathbb{R}^n) \mid f(x) \geq 0 \ a.e. \right\}.$$

  (a) Show that K is a closed convex cone.
  (b) Show that K has empty interior.
  (c) Given $f \in L^2(\mathbb{R}^n)$, what is $P_K(f)$, i.e., the projection of f onto K.
  (d) Given $f \in L^2(\mathbb{R}^n) \setminus K$, is it possible to properly separate f from K? If so, provide a properly separating hyperplane, if not, provide a counter-example.