Math 555 Winter 2015
Homework Set 6 **Due Monday, February 23**

(1) A collection $A_1, A_2, \ldots$ of measurable subset of $\mathbb{R}^n$ is said to be *almost disjoint* if

$$\lambda(A_j \cap A_k) = 0 \qquad \text{for} \quad j \neq k \ .$$

(a) Prove that if $A_1, A_2, \ldots$ are almost disjoint, then

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda(A_k) \ .$$

(b) Conversely, suppose that the measurable sets $A_1, A_2, \ldots$ satisfy

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda(A_k) < \infty \ .$$

Show that the sets $A_1, A_2, \ldots$ are almost disjoint. Does this remain true if the "$< \infty$" is replaced by the weaker hypothesis that $\lambda(A_k) < \infty$ for each $k = 1, 2, \ldots$?

(c) Suppose $A_1, A_2, \ldots$ are measurable sets, and suppose that there is a positive integer d such that each point $x \in \mathbb{R}^n$ belongs to no more than d of the A_k's. Prove that

$$\sum_{k=1}^{\infty} \lambda(A_k) \leq d\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) \ .$$

(2) Define the function on $(0,1)$ as follows: given $x \in (0,1)$, let x have decimal expansion $0.a_1a_2a_3\ldots$ where each $a_i \in \{0,1,\ldots,9\}$ and the expansion is non-repeating whenever possible (i.e. $0.5 = 0.49999\ldots$) to ensure uniqueness, then set

$$f(0.a_1a_2a_3\ldots) = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is the smallest integer for which } a_n = 7\text{,and} \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is measurable. What is $\int_0^1) f(x)dx$?
Hint: The Taylor series for $\ln(1-x)$ can be useful here.

(3) (a) Show that the function $\frac{\sin x}{x}$ is not integrable on $(0,\infty)$.

(b) Show that $\lim_{R\to\infty} \int_0^R \frac{\sin x}{x} dx$ exists.

(c) Show that if $f \geq 0$ is measurable on $(0,\infty)$, then

$$\lim_{R\to\infty} \int_0^R f(x)dx < \infty \quad \Longleftrightarrow \quad f \text{ is integrable on } (0,\infty).$$

(4) For the function

$$f(x,y) = \frac{x-y}{(x+y)^3},$$

show that the iterated integrals

$$\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx \quad \text{and} \quad \int_0^1 \left(\int_0^1 f(x,y)dx\right) dy$$

both exist, but are not equal. Show that this does not violate Fubini's Theorem by showing that

$$\int_{[0,1]\times[0,1]} |f(x,y)| dx dy = \infty \ .$$

(5) Let $f : [0,1] \mapsto \mathbb{R}$ be a non-negative, continuous, and strictly increasing function on $[0,1]$. The the inverse function f^{-1} exists and is continuous on $[f(0), f(1)]$. Define

$$\mathcal{R}\int_0^1 f(x)dx = \lim_{N\to\infty} 2^{-N} \sum_{k=1}^{2^N} f(k2^{-N}) \qquad \text{(the Riemann integral)}$$

and

$$\mathcal{L}\int_0^1 f(x)dx = \lim_{N\to\infty} \sum_{k=1}^{2^N} y_k \left(f^{-1}(y_k) - f^{-1}(y_{k-1})\right) \qquad \text{(the Lebesgue integral)}$$

where

$$y_k = f(0) + k2^{-N}\left(f(1) - f(0)\right) .$$

(a) Prove directly that both limits exist.

(b) Prove that

$$\mathcal{R}\int_0^1 f(x)dx = \mathcal{L}\int_0^1 f(x)dx \ .$$

For this you may quote a theorem from the theory of Riemann integration, but give a precise statement and reference.