(1) A collection $A_1, A_2, \ldots$ of measurable subset of $\mathbb{R}^n$ is said to be *almost disjoint* if
\[ \lambda(A_j \cap A_j) = 0 \quad \text{for} \quad j \neq k. \]
(a) Prove that if $A_1, A_2, \ldots$ are almost disjoint, then
\[ \lambda \left( \bigcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \lambda(A_k). \]
(b) Conversely, suppose that the measurable sets $A_1, A_2, \ldots$ satisfy
\[ \lambda \left( \bigcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \lambda(A_k) < \infty. \]
Show that the sets $A_1, A_2, \ldots$ are almost disjoint. Does this remain true if the "$< \infty$" is replaced by the weaker hypothesis that $\lambda(A_k) < \infty$ for each $k = 1, 2, \ldots$?
(c) Suppose $A_1, A_2, \ldots$ are measurable sets, and suppose that there is a positive integer $d$ such that each point $x \in \mathbb{R}^n$ belongs to no more than $d$ of the $A_k$’s. Prove that
\[ \sum_{k=1}^{\infty} \lambda(A_k) \leq d \lambda \left( \bigcup_{k=1}^{\infty} A_k \right). \]
(2) Define the function on $(0, 1)$ as follows: given $x \in (0, 1)$, let $x$ have decimal expansion $0.a_1a_2a_3 \ldots$ where each $a_i \in \{0, 1, \ldots, 9\}$ and the expansion is non-repeating whenever possible (i.e. 0.5 = 0.49999…) to ensure uniqueness, then set
\[ f(0.a_1a_2a_3 \ldots) = \begin{cases} \frac{1}{n}, & \text{if $n$ is the smallest integer for which } a_n = 7, \text{and} \\ 0, & \text{otherwise.} \end{cases} \]
Show that $f$ is measurable. What is $\int_{0}^{1} f(x)dx$?
Hint: The Taylor series for $\ln(1 - x)$ can be useful here.
(3) (a) Show that the function $\frac{\sin x}{x}$ is not integrable on $(0, \infty)$.
(b) Show that $\lim_{R \to \infty} \int_{0}^{R} \frac{\sin x}{x} dx$ exists.
(c) Show that if $f \geq 0$ is measurable on $(0, \infty)$, then
\[ \lim_{R \to \infty} \int_{0}^{R} f(x) dx < \infty \iff f \text{ is integrable on } (0, \infty). \]
(4) Evaluate $\int_{-\infty}^{\infty} e^{-x^2} dx$ by computing
\[ \int_{\mathbb{R}^{2}} e^{-(x^2+y^2)} dxdy \]
in two ways:
(a) by directly using Fubini’s Theorem, and
(b) by using polar coordinates in $\mathbb{R}^{2}$. 
(5) For the function 
\[ f(x, y) = \frac{x - y}{(x + y)^3}, \]
show that the iterated integrals
\[ \int_0^1 \left( \int_0^1 f(x, y) dy \right) dx \quad \text{and} \quad \int_0^1 \left( \int_0^1 f(x, y) dx \right) dy \]
both exist, but are not equal. Show that this does not violate Fubini’s Theorem by showing that
\[ \int_{[0,1] \times [0,1]} |f(x, y)| dxdy = \infty. \]

(6) Let \( f : [0, 1] \to \mathbb{R} \) be a non-negative, continuous, and strictly increasing function on \([0, 1]\). The the inverse function \( f^{-1} \) exists and is continuous on \([f(0), f(1)]\). Define
\[ \mathcal{R} \int_0^1 f(x)dx = \lim_{N \to \infty} 2^{-N} \sum_{k=1}^{2^N} f(k2^{-N}) \quad \text{(the Riemann integral)} \]
and
\[ \mathcal{L} \int_0^1 f(x)dx = \lim_{N \to \infty} \sum_{k=1}^{2^N} y_k \left( f^{-1}(y_k) - f^{-1}(y_{k-1}) \right) \quad \text{(the Lebesgue integral)} \]
where
\[ k_k = f(0) + k2^{-N} (f(1) - f(0)). \]

(a) Prove directly that both limits exist.
(b) Prove that
\[ \mathcal{R} \int_0^1 f(x)dx = \mathcal{L} \int_0^1 f(x)dx. \]
For this you may quote a theorem from the theory of Riemann integration, but give a precise statement and reference.