Math 555 — Winter 2015

Homework 5 — **Due Friday, February 6**

(1) For scalar equations ($n = 1$), explicitly determine the formula for the Taylor method of order 3.

(2) Heun's third-order method is the 3-stage Runge–Kutta method:

$$x_{i+1} = x_i + \tfrac{h}{4}(k_1 + 3k_3),$$

where

$$\begin{aligned} k_1 &= f(t_i, x_i), \\ k_2 &= f(t_i + \tfrac{h}{3},\ x_i + \tfrac{h}{3}k_1), \quad \text{and} \\ k_3 &= f(t_i + \tfrac{2}{3}h,\ x_i + \tfrac{2}{3}hk_2)\ . \end{aligned}$$

Show that this is accurate of order 3 (for $n = 1$).

(3) Show that every explicit Runge-Kutta method is stable. That is, if $f(t, x)$ is continuous in (t, x) and uniformly Lipschitz continuous in x, then $\psi(h, t, x)$ is continuous in (h, t, x) and uniformly Lipschitz continuous in x.

(4) Derive an implicit 2-step scheme for the equation $x' = f(t, x)$ by approximating the integral in

$$x(t_{i+2}) = x(t_i) + \int_{t_i}^{t_{i+2}} f(t, x(t))dt$$

using Simpson's rule. What is the order of accuracy of your scheme?

(5) Consider the linear multi-step method (for a constant b)

$$x_{i+2} + (b-1)x_{i+1} - bx_i = \frac{h}{4}\left[(b+3)f_{i+2} + (3b+1)f_i\right]\ .$$

(a) Show that this method is accurate of order 2 if $b \neq -1$ and accurate of order 3 if $b = -1$.

(b) Show that the method is *not* zero-stable if $b = -1$. For what values of b is it zero-stable?

(c) Illustrate the resulting divergence of the method with $b = -1$ by applying it to the IVP $x' = x,\ x(0) = 1$, and solving exactly the resulting difference equation when the starting values are $x_0 = 1,\ x_1 = 1$.

(6) Show that if r is a root of multiplicity $m > 1$ of the characteristic polynomial of the linear difference equation

$$x_{i+k} + a_{k-1}x_{i+k-1} + \ \ldots\ + a_1x_{i+1} + a_0x_i = 0,$$

then the sequences $x_i = i^j r^i$ for $0 \le j \le m-1$ are solutions.