(1) We introduce a notion of differentiability for real valued functions on $\mathbb{C}$ by defining a corresponding function on $\mathbb{R}^2$. Define the linear mapping $\Theta : \mathbb{R}^2 \to \mathbb{C}$ by $\Theta(x_1, x_2) = x_1 + \mathrm{i}x_2$. Then, given $f : \mathbb{C} \to \mathbb{R}$, define a corresponding mapping $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ by $\tilde{f} = f \circ \Theta$. We say that the mapping f is *differentiable in the real sense* if $\tilde{f}$ is differentiable, and f is *twice differentiable in the real sense* if $\tilde{f}$ is twice differentiable. The chain rule gives $f'(\zeta) = \Theta \nabla \tilde{f}(\Theta^{-1}\zeta)$ and $f''(\zeta)\delta = \Theta \nabla^2 \tilde{f}(\Theta^{-1}\zeta)\Theta^{-1}\delta$.

Differentiability in the real sense is the only notion of differentiability we use for mappings from $\mathbb{C}$ to $\mathbb{R}$, so when we say f is differentiable it means f is differentiable in the real sense. The directional derivative of f in the direction δ is given by $f'(\zeta; \delta) = \mathrm{Re}(\overline{f'(\zeta)}\delta)$, and the second derivative is given by $f''(\zeta; \omega, \delta) = \mathrm{Re}(\bar{\omega} f''(\zeta)\delta)$. The function f is quadratic if $f''(\zeta)$ is constant in ζ. For example, the function $r_2(\zeta) = |\zeta|^2/2$ is quadratic with $r_2'(\zeta) = \zeta$ and $r_2''(\zeta) = I$.

(a) Suppose $f : \mathbb{C} \to \mathbb{R}$ is differentiable on $\mathbb{C}$, show that

$$f(y) = f(x) + f'(x; y - x) + o(|y - x|).$$

(b) This notion of differentiability is easily extended to mappings $f : \mathbb{C}^n \to \mathbb{R}$. Describe how this is done, and define a notion of the gradient, $\nabla f(x)$, that works with this notion of differentiability (hint: you should get $f'(x; d) = \mathrm{Re}(\nabla f(x)^H d) = \mathrm{Re}\,\langle \nabla f(x), d\rangle$). Then use these definitions to show that

$$f(y) = f(x) + f'(x; y - x) + o(|y - x|).$$

(c) We have the following theorem from the course notes:

Theorem *Suppose that the autonomous DE $x' = f(x)$ (with $f \in C^1$) has a Lyapunov function V for which*

$$\mathrm{Re}(\nabla V(x)^H f(x)) < 0 \quad \textit{whenever } x \neq 0 \textit{ with } \mathrm{Re}(\nabla V(x)^H f(x)) = 0 \textit{ when } x = 0.$$

Then every solution to DE is asymptotically stable to zero.

What goes wrong in the non-autonomous case $x' = f(t, x)$? Are there reasonable additional conditions that one might impose on $f(t, x)$ for which one can prove a non-autonomous version of the theorem stated above? If so, state and prove such a result.

(2) Prove the following lemma.

Lemma *The mapping $\mathcal{H} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ given by $\mathcal{H}(M) = \frac{1}{2}(M + M^*)$ is the orthogonal projection of $\mathbb{C}^{n\times n}$ onto the subspace $\mathbb{H}^n$ in the Frobenius inner product. Similarly, the mapping $\mathcal{S} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by $\mathcal{S}(M) = \frac{1}{2}(M + M^T)$ is the orthogonal projection of $\mathbb{R}^{n\times n}$ onto the subspace $\mathbb{S}^n$.*

(3) Let $M \in \mathbb{H}^n$ and show that $M \preceq \gamma I$ if and only if $\lambda \leq \gamma$ for all $\lambda \in \sigma(M)$.

(4) Prove the following theorem.

Theorem *[Exponential Stability of (CLH)] Let $A \in \mathbb{C}^{n\times n}$ be such that $\alpha(A) < 0$. Then, given $\alpha(A) < \gamma < 0$, there exists $0 \prec P \in \mathbb{H}^n$ such that*

$$PA + A^H P \preceq 2\gamma P.$$

Moreover, for every solution $x(t)$ to (CLH),

$$|x(t)| \leq \sqrt{\kappa(P)}\,|x(t_0)|e^{\gamma(t-t_0)} \qquad \forall\ t \geq t_0,$$

where $\kappa(P)$ is the condition number for the matrix P.

(5) Let $0 < \epsilon << 1$ and consider the matrix

$$A = \begin{bmatrix} -\epsilon & \epsilon^{-2} \\ 0 & -\epsilon \end{bmatrix}.$$

(a) Obtain a closed form expression for the solution $x(t)$ for the IVP $x' = Ax$ with $x(0) = \delta(1,1)^T$ for $\delta > 0$, and use Matlab to graph this solution for $\epsilon = 0.1$ and $\delta = 1$.

(b) Use this A to show that the stability bound in the theorem of problem (4) is sharp, i.e. one cannot do better than $\sqrt{\kappa(P)}$ and obtain an exponential stability bound (hint: consider the solution near time $t = \epsilon^{-1} - \epsilon^2$).