Math 555 Winter 2015
Homework 3 **Due Friday, January 23**

(1) Let $A = \begin{bmatrix} 10 & 6 \\ -18 & -11 \end{bmatrix}$

(a) Compute e^{tA}.
(b) Find the general solution of $x' = Ax$.

(2) Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Compute a fundamental matrix for the system $x' = Ax$.
(b) Suppose $f : \mathbb{R} \to \mathbb{R}^3$ is continuous and

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty.$$

Show that all solutions of $x' = Ax + f$ stay bounded as $t \to +\infty$.

(3) Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ be continuous and consider the matrix function

$$\Phi(t) = \exp\left(\int_0^t A(s) ds\right).$$

(a) Observe that Φ is a fundamental matrix for the system $x' = A(t)x$ if either $n = 1$ or A is constant.
(b) Give an example for which Φ is not a fundamental matrix for $x' = A(t)x$.
(c) Provide, with proof, a general condition (for a non-constant A and general n) that guarantees that Φ is a fundamental matrix.
(d) Solve the IVP

$$x' = A(t)x + b(t), \qquad x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where

$$b(t) = \begin{bmatrix} t \\ t \end{bmatrix} \quad \text{and} \quad A(t) = \begin{bmatrix} t & t^3 \\ 0 & t \end{bmatrix}.$$

(4) Let $A : \mathbb{R} \to \mathbb{C}^{n\times n}$ be continuous and suppose that

$$\liminf_{t\to\infty} \operatorname{Re} \int_0^t \operatorname{tr}(A(s)) ds > -\infty .$$

Suppose that $\Phi(t)$ is a fundamental matrix for the system $x' = A(t)x$, and suppose that Φ is uniformly bounded on $[0, \infty)$ (in some norm).

(a) Show that Φ^{-1} is uniformly bounded on $[0, \infty)$.
(b) Show that no non-trivial solution to $x' = A(t)x$ can satisfy $x(t) \to 0$ as $t \to \infty$.

(5) Let $A \in \mathbb{C}^{n\times n}$. Use the Spectral Mapping Theorem to prove that $\det(e^A) = e^{\operatorname{tr}(A)}$.