Math 555 Winter 2015

Homework Set 2 **Due Friday January 16**

(1) Given a function $p(t,x,\xi) \in C^2([a,b]\times\mathbb{R}^n\times\mathbb{R}^n)$, the Hamiltonian flow for p is the system in (x,ξ):

$$x'(t) = (\nabla_\xi p)(t,x(t),\xi(t)), \quad \xi'(t) = -(\nabla_x p)(t,x(t),\xi(t)).$$

(a) Use the change of variables formula to show that the corresponding flow maps $S_{t_0}^{t_1} : \mathbb{R}^{2n}\times\mathbb{R}^{2n}$ preserve measure; that is,

$$\int f(S_{t_0}^{t_1}(x,\xi))\, dx\, d\xi \;=\; \int f(x,\xi)\, dx\, d\xi\ ,$$

where you may take f to be continuous and supported in a compact set.

(b) Show that $\partial_t(p(t,x(t),\xi(t))) = (\partial_t p)(t,x(t),\xi(t))$. Conclude that if p is autonomous, then $p(x(t),\xi(t))$ is constant in t.

(c) For a body falling on a straight line towards the earth, with x the distance from the center of the earth, and ξ the velocity of the object (negative is downwards) then (x,ξ) satisfies the Hamiltonian flow with

$$p(x,\xi) = \frac{1}{2}\xi^2 - \frac{gM}{x},$$

where g is the gravitational constant and M is the mass of the earth. Use part (b) to derive an equation $\frac{d\xi}{dt} = f(\xi)$ for an object dropped from initial position $x = R$ and initial velocity 0.

(2) Suppose that $A(t)$ is a continuous function from an interval I into $\mathbb{C}^{n\times n}$ such that $A(t)$ skew symmetric for all $t\in I$ $(A(t) = -A^*(t))$.

(a) If $\Phi(t)$ is a fundamental matrix for $x' = Ax$ on I normalized at $t_0 \in I$, show that $\Phi(t)$ is unitary for all $t \in I$. In particular, if $A(t)$ is real valued, then $\Phi(t)$ is an orthogonal matrix.

(b) In general, if $Q\in\mathbb{R}^{n\times n}$, and $Q(v,w)$ is the quadratic form on $\mathbb{C}^n$ given by

$$Q(v,w) = \sum_{j=1}^{n} (Qv)_j \bar{w}_j,$$

show that $Q(\Phi(t)v,\Phi(t)w) = Q(v,w)$ for all vectors v and w provided that

$$A^*(t)Q + QA(t) = 0 \quad \forall\, t\in I.$$

(c) On an even dimensional space $\mathbb{R}^{2n}$, let Q be the matrix in block form

$$Q = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

The corresponsing skew-symmetric form on $\mathbb{R}^{2n}$ is the *symplectic form*, i.e.,

$$Q\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} y \\ x \end{pmatrix},$$

and a matrix M preserving this form (i.e. $M^TQM = Q$) is called a *symplectic matrix*. Expressing $A(t)$ in block form

$$A(t) = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix},$$

where the blocks are $n \times n$ derive a condition on the $A_j(t)$ such that the above holds; that is, so that $\Phi(t)$ is symplectic. Show that the linearized Hamiltonian flow matrix of problem 1 above is symplectic. That is, $\frac{D(x,\xi)}{D(y,\eta)}$ is symplectic, for each (y,η) and t, where $x(t_0) = y$, $\xi(t_0) = \eta$ and $(x,\xi) \in \mathbb{R}^{2n}$.

Remark: For background on symplectic forms see
`https://web.math.princeton.edu/~templier/junior/5/Symplectic_Matrices_Talk.pdf`

(3) Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \mathbb{C}$ be an eigenvalue of A of algebraic multiplicity 2 but geometric multiplicity 1. If $v \in \mathbb{C}^n$ is an eigenvector with eigenvalue λ, then $e^{\lambda t}v$ is a solution of $x' = Ax$. We want to find a second linearly independent solution corresponding to the eigenvalue.

(a) Try another solution of the form
$$x(t) := e^{\lambda t}(ty + z) \quad \text{with } y, z \in \mathbb{C}^n.$$
Derive conditions on (y,z) in order that this x solves the system. Show that one can always find (y,z) to obtain a solution of this form.

(b) Find a fundamental set of solutions for the sytem $x' = Ax$ with
$$A = \begin{bmatrix} 8 & 4 \\ -9 & -4 \end{bmatrix}.$$

(c) Find an explicit similarity transformation to put A from part (b) into Jordan form.

(4) Let A be the matrix
$$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & -2 \end{bmatrix}.$$

(a) Find the Jordan decomposition of A.

(b) Find a fundamental matrix for the initial value problem $x' = Ax$, $x(0) = x_0$.

(c) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous vector-valued function with $\int_0^\infty |f(t)|dt < \infty$. Let $x(t)$ be the solution to the following initial value problem for $t \geq 0$:
$$x'(t) = Ax(t) + f(t), \quad x(0) = 0.$$
Show there exists a constant vector $v \in \mathbb{R}^3$ such that $|x(t) - v| \to 0$ as $t \to +\infty$, and calculate v in terms of f (Your answer is allowed to involve the inverse of a matrix.)