Math 555 Winter 2006

Homework Set 1 **Due Friday January 9**

(1) Osgood's Uniqueness Theorem: Suppose that $\phi : \mathbb{R}_+ \mapsto \mathbb{R}$ is a continuous increasing positive function on $\mathbb{R}_+$ for which $\int_0^1 \frac{du}{\phi(u)} = \infty$. Show that if $f : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuous and $|f(t,x) - f(t,y)| \leq \phi(|x-y|)$, then the IVP $x' = f(t,x),\ x(0) = x_0$ has at most one solution.

(2) Let $\mathcal{D} \subset \mathbb{R}^2$ be open and nonempty, and suppose that $f : \mathcal{D} \to \mathbb{R}$ is continuous. Let $(t_0, x_0) \in \mathcal{D}$. Show that if there exits $\beta > 0$ and convex set $\Omega \subset \mathbb{R}$ with $[t_0, t_0+\beta]\times\Omega \subset \mathcal{D}$ such that the initial value problem

$$IVP: \quad x' = f(t,x), \quad x(t_0) = x_0$$

has two distinct solutions $x_1(t)$ and $x_2(t)$ with $x_i(t) \in \Omega$ for all $t \in [t_0, t_0 + \beta]$, i=1,2, then there are infinitely many solutions to the initials value problem IVT on I.

(3) Let $n = 1,\ \mathbb{F} = \mathbb{R}$. Let $f(u)$ be a positive continuous function on $[u_0, \infty)$. Consider the IVP $u' = f(u),\ u(0) = u_0$..

(a) Use the inverse function theorem to give a *rigorous* justification of the method of *separation of variables* to solve this problem by proving that the equation

$$\int_{u_0}^{u} \frac{dv}{f(v)} = t$$

determines a $\mathcal{C}^1$ function $u(t)$ that is the unique solution of the IVP for $t \geq 0$.

(b) Show that the solution of this IVP exists for all time $t \geq 0$ if and only if $\int_{u_0}^{\infty} \frac{dv}{f(v)} = \infty$.

(4) Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ and $f : \mathbb{R}^N \mapsto \mathbb{R}^N$ be $\mathcal{C}^1$. Suppose that

$$\nabla V(x) \cdot f(x) \leq 0 \quad \text{for all } x \in \mathbb{R}^n .$$

(a) Interpret this inequality geometrically.

(b) Further, suppose that there is an $\alpha > 0$ such that

$$V(x) \geq \alpha |x|^2 \quad \text{for all } x \in \mathbb{R}^n .$$

Show that for any $x_0 \in \mathbb{R}^n$, the solution of the IVP $x' = f(x),\ x(0) = x_0$ can be extended to all of $[0, \infty)$.

Remark: $V(x)$ is called a *Liapunov function* for $f(x)$.