The first three problems deal with uniqueness theorems having weaker hypotheses than the Lipschitz condition.

(1) One-sided uniqueness theorem \((n = 1, \mathbb{F} = \mathbb{R})\)

(a) A real-valued function \(f(t, u)\) is said to satisfy a one-sided Lipschitz condition in \(u\) if

\[ u_1 > u_2 \implies f(t, u_1) - f(t, u_2) \leq L(u_2 - u_1) \quad \forall \ t \in \mathbb{R}. \]

Show that if \(f\) is continuous in \(t\) and \(u\) and satisfies a one-sided Lipschitz condition in \(u\), then there is at most one solution to the IVP \(u' = f(t, u), \ u(t_0) = 0\), for \(t \geq t_0\).

(b) Let \(f(t, u)\) be a real-valued continuous function in \(t\) and \(u\), and suppose that \(f\) is decreasing in \(u\) for all \(t\), i.e., \(u_2 > u_1\) implies that \(f(t, u_2) \leq f(t, u_1)\). Show that if \(u(t)\) and \(v(t)\) are both solutions to \(u' = f(t, u)\), then

\[ |u(t) - v(t)| \leq |u(s) - v(s)| \quad \text{whenever} \quad t \geq s. \]

Deduce uniqueness for the IVP \(u' = f(t, u), \ u(t_0) = 0\), for \(t \geq t_0\). Show, however, that uniqueness may fail for \(t < t_0\).

(2) Let \(f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n\) be continuous and satisfy the *generalized Lipschitz condition*

\[ |f(t, x) - f(t, y)| \leq \kappa(t)|x - y| \quad \forall \ t \in [0, a] \text{ and } x, y \in \mathbb{R}^n, \]

where \(\kappa(t)\) is non-negative and continuous on \((0, a]\), but possibly unbounded at \(t = 0\). Show that if \(\int_0^a \kappa(t)dt < \infty\), then the IVP \(x' = f(t, x), \ x(0) = x_0\), has at most one solution on \([0, a]\).

(3) Osgood’s Uniqueness Theorem: Suppose that \(\phi : \mathbb{R}_+ \to \mathbb{R}\) is a continuous increasing positive function on \(\mathbb{R}_+\) for which \(\int_0^1 \frac{dt}{\phi(t)} = \infty\). Show that if \(f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\) is continuous and \(|f(t, x) - f(t, y)| \leq \phi(|x - y|)\), then the IVP \(x' = f(t, x), \ x(0) = x_0\) has at most one solution.

(4) Let \(n = 1, \mathbb{F} = \mathbb{R}\). Suppose \(f : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}\) satisfies a Lipschitz condition for \(t \geq t_0\). Further suppose that \(u(t)\) satisfies the differential inequality \(u' \leq f(t, u)\) for \(t \geq t_0\), and \(v(t)\) satisfies \(v' = f(t, v)\) for \(t \geq t_0\). Show that if \(u(t_0) < v(t_0)\), then \(u(t) < v(t)\) for all \(t \geq t_0\).

(5) Show that if there are two distinct solutions to \(u' = f(t, u)\) \((n = 1, \mathbb{F} = \mathbb{R})\) satisfying the same initial conditions at \(t_0\), then there are infinitely many solutions satisfying the same initial condition at \(t_0\).

(6) Let \(f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n\) be continuous and satisfy the generalized Lipschitz condition

\[ |f(t, x) - f(t, y)| \leq L(t)|x - y|, \]

with \(L : [0, \infty) \to [0, \infty)\) continuous and satisfying \(\int_0^\infty L(t)dt < \infty\). Show that the IVP \(x' = f(t, x), \ x(t_0) = x_0\) (where \(0 \leq t_0\)) has a solution on \([0, \infty)\). In addition, show that if one such solution is bounded, then all solutions must be bounded.