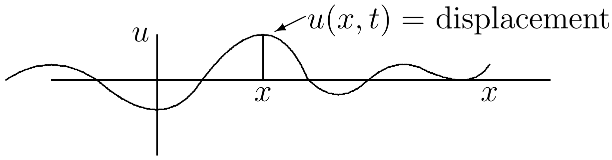
Vibrating Strings and Heat Flow

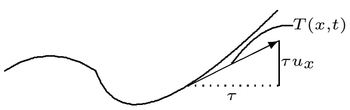
Consider an infinite vibrating string. Assume that the x-axis is the equilibrium position of the string and that the tension in the string at rest in equilibrium is τ. Let $u(x,t)$ denote the displacement at x at time t. Then the *wave equation* (in one space dimension) governs the motion.

“snap shot” at time t:



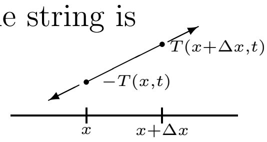
Derivation. (for small displacements). We make the following simplifying assumptions:

- the displacement of the string from equilibrium (and its slope) are small;
- each point on the string moves only in the vertical direction;
- the tension force $T(x,t)$ in the string (i.e., the (vector) force which the part of the string to the right of x exerts on the part to the left of x, at time t) is tangential to the string and has magnitude proportional to the local stretching factor $\sqrt{1+u_x^2}$.



Since $u_x = 0$ in equilibrium, the constant of proportionality is the equilibrium tension τ. Thus the magnitude of $T(x,t)$ is $\tau\sqrt{1+u_x(x,t)^2}$, and the vertical component of $T(x,t)$ is τu_x. Now consider the part of the string between x and $x+\Delta x$. The vertical component of Newton's second law (force = mass × acceleration) applied to this part of the string is

$$\overbrace{\tau u_x(x+\Delta x,t)-\tau u_x(x,t)}^{\text{force}} = \overbrace{\rho\Delta x}^{\text{mass}}\,\overbrace{u_{tt}(x,t)}^{\text{accel}},$$



where ρ is the density (mass per unit length; assumed constant). Dividing by Δx and taking the limit as $\Delta x \to 0$, we obtain

$$\tau u_{xx} = \rho u_{tt}.$$

Normalizing units so that $\rho = \tau$, we obtain the *wave equation* (in one space dimension):

$$u_{tt} = u_{xx}.$$

Solutions of $u_{tt} = u_{xx}$

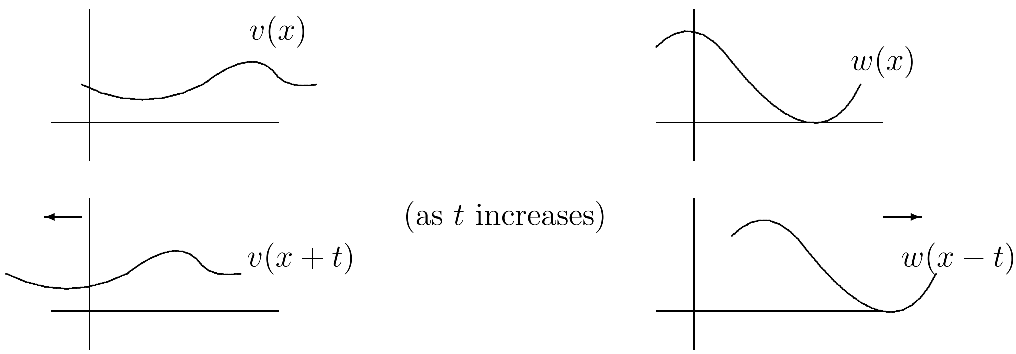
Change variables. Let $y = x + t$, $z = x - t$, so $x = \frac{1}{2}(y+z)$, $t = \frac{1}{2}(y-z)$. Then

$$\begin{aligned} \partial_y &= (\partial_y x)\,\partial_x + (\partial_y t)\,\partial_t \;=\; \tfrac{1}{2}\,(\partial_x + \partial_t)\,, \\ \partial_z &= (\partial_z x)\partial_x + (\partial_z t)\partial_t \;=\; \tfrac{1}{2}\,(\partial_x - \partial_t)\,, \end{aligned}$$

so $u_y = \frac{1}{2}(u_x + u_t)$ and $u_{yz} = \frac{1}{2}\,(\partial_x - \partial_t)\,\frac{1}{2}(u_x + u_t) = \frac{1}{4}(u_{xx} - u_{tt})$. In the new coordinates, the wave equation becomes simply $u_{yz} = 0$. Thus u_y is independent of z, i.e., $u_y = \widetilde{v}(y)$. Integrating in y for each fixed z, we get $u = v(y) + w(z)$ (where $v(y) = \int \widetilde{v}(y)dy$). So any solution of the wave equation $u_{tt} = u_{xx}$ is of the form

$$(*) \qquad u(x,t) = v(x+t) + w(x-t).$$

Physically, this is a superposition of left-going and right-going waves:



Observation. The derivation above shows that any C^2 function of x and t satisfying the wave equation is of the form $(*)$. Conversely, if v and w are C^2 functions of one variable, it is easily checked that $u(x,t) = v(x+t) + w(x-t)$ is a C^2 solution of the wave equation. But if v and w are only continuous, $v(x+t) + w(x-t)$ still makes sense; in what sense is this a solution of $u_{tt} = u_{xx}$? We will see later in the course that the equation holds in the sense of distributions.

Initial-Value Problem (IVP) (or the *Cauchy Problem*)

If we think of the wave operator as an ordinary differential operator in time acting on functions of t taking values in functions of x (overlooking considerations arising from the fact that ∂_x^2 is itself a differential operator), we "should" be able to determine $u(x,t)$ for $x \in \mathbb{R}$ and $t \geq 0$ if we are given initial values $u(x,0)$ and $u_t(x,0)$ for $x \in \mathbb{R}$ (we need u and u_t at $t = 0$ since the equation is second-order in t).

D'Alembert's Formula (for the Cauchy Problem for $u_{tt} = u_{xx}$)

Consider the IVP: DE $u_{tt} = u_{xx}$ $(x \in \mathbb{R}, t \geq 0)$

$$\text{IC}\begin{cases} u(x,0) = f(x) & (x \in \mathbb{R}) \\ u_t(x,0) = g(x) & (x \in \mathbb{R}) \end{cases}$$

(To obtain a C^2 solution $u(x,t)$, it will suffice for $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$.) We will separately analyze the cases $g \equiv 0$ and $f \equiv 0$, and then use superposition.

Case 1. $g \equiv 0$.

$$\text{IC}\begin{cases} u(x,0) &= f(x) \\ u_t(x,0) &= 0 \end{cases} \quad (x \in \mathbb{R}).$$

We have $u(x,t) = v(x+t) + w(x-t)$ for some $v, w \in C^2(\mathbb{R})$. By the IC,

$$\begin{aligned} v(x) + w(x) &= u(x,0) \;=\; f(x) \\ v'(x) - w'(x) &= u_t(x,0) \;=\; 0, \end{aligned}$$

so v and w differ by a constant. One solution is $v(x) = w(x) = \frac{1}{2}f(x)$. Any other solution is $\begin{aligned} v(x) &= \tfrac{1}{2}f(x) + c \\ w(x) &= \tfrac{1}{2}f(x) - c \end{aligned}$ for some constant c. So the solution in Case 1 is

$$u(x,t) = \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t).$$

Remark. For a solution $u(x,t)$ of $u_{tt} = u_{xx}$, v and w are uniquely determined up to a constant. This is because if $v_1(x+t) + w_1(x-t) = v_2(x+t) + w_2(x-t)$, then $v_1(x+t) - v_2(x+t) = w_2(x-t) - w_1(x-t)$ is independent of both $y = x+t$ and $z = x-t$, and is thus a constant.

Case 2. $f \equiv 0$.

$$\text{IC}\begin{cases} u(x,0) &= 0 \\ u_t(x,0) &= g(x) \end{cases} \quad (x \in \mathbb{R}).$$

Again, $u(x,t) = v(x+t) + w(x-t)$ for some $v, w \in C^2(\mathbb{R})$. By the IC,

$$\begin{aligned} v(x) + w(x) &= 0 \\ v'(x) - w'(x) &= g(x) \end{aligned}. \qquad \text{Thus} \qquad \begin{aligned} w &= -v \\ v' &= \tfrac{1}{2}g \end{aligned}.$$

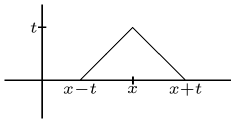
So $v = \frac{1}{2}\int g$ and the solution in Case 2 is

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)ds.$$

Adding Cases 1 and 2, the solution of the IVP with IC $\begin{cases} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{cases}$ is given by d'Alembert's formula:

$$u(x,t) = \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds.$$

Remarks.

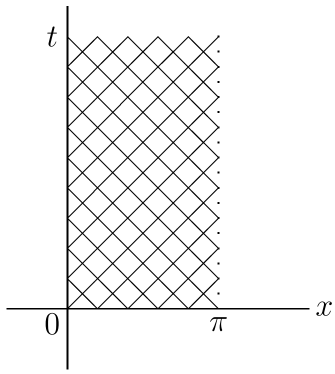


(1) d'Alembert's formula gives an explicit demonstration of the *finite domain of dependence* of the solution of this IVP on the initial data (a general property of hyperbolic PDEs): for a fixed $x \in \mathbb{R}$ and fixed $t > 0$, $u(x,t)$ depends only on $f(x+t)$, $f(x-t)$, and $\{g(s) : x-t \le s \le x+t\}$.

(2) d'Alembert's formula also provides a solution for negative t as well: $u_{tt} = u_{xx}$ $(x \in \mathbb{R}, t \le 0)$, $\begin{cases} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{cases}$ ("final" conditions); like ODEs, hyperbolic PDEs in general can be advanced either in the $+t$ direction or the $-t$ direction.

Initial-Boundary Value Problem (IBVP)

Consider now a finite string ($0 \le x \le \pi$) fixed at both ends, so $u(0,t) = u(\pi,t) \equiv 0$. Suppose the initial displacement is $u(x,0) = f(x)$ $(0 \le x \le \pi)$ (where $f(0) = f(\pi) = 0$), and for simplicity suppose the initial velocity is $u_t(x,0) = 0$ $(0 \le x \le \pi)$. This models a "plucked" violin string (moved to position $u(x,0) = f(x)$ at time $t = 0$, and then released with initial velocity $u_t(x,0) = 0$). We obtain an IBVP with both initial conditions (IC) and boundary conditions (BC):



$$\text{DE} \quad u_{tt} = u_{xx} \qquad (0 \le x \le \pi, t \ge 0)$$

$$\text{IC} \quad \begin{cases} u(x,0) &= f(x) \\ u_t(x,0) &= 0 \end{cases} \qquad (0 \le x \le \pi)$$

$$\text{BC} \quad \begin{cases} u(0,t) &= 0 \\ u(\pi,t) &= 0 \end{cases} \qquad (t \ge 0)$$

We will solve this IBVP in two ways: ① by d'Alembert's formula, and ② by Fourier series.

Solution ① (d'Alembert). Find functions v, w defined on $\mathbb{R}$ so that

$$u(x,t) = v(x+t) + w(x-t)$$

satisfies the IC and BC. The BC $u(0,t) = 0$ for $t \ge 0$ gives $0 = v(t) + w(-t)$ for $t \ge 0$, or $w(t) = -v(-t)$ for $t \le 0$. [Note that to define $u(x,t)$ in the region $0 \le x \le \pi$, $t \ge 0$, we only need to give $v(s)$ for $s \ge 0$ and $w(s)$ for $s \le \pi$. To simplify our calculations, we will find v and w defined on all of $\mathbb{R}$, so that $u(x,t)$ satisfies the BC for $t \le 0$ too.] So we ask $w(t) = -v(-t)(\forall\, t \in \mathbb{R})$. Next, the BC $u(\pi,t) = 0$ (now $\forall\, t \in \mathbb{R}$) gives $0 = v(\pi+t)+w(\pi-t)$, so $v(\pi+t) = -w(\pi-t) = v(t-\pi)$, i.e., $v(t+2\pi) = v(t)(\forall\, t \in \mathbb{R})$. So v is 2π-periodic, and thus $w(t) = -v(-t)$ is also 2π-periodic. The IC $u_t(x,0) = 0$ $(0 \le x \le \pi)$ gives

$0 = v'(x) - w'(x) = v'(x) - v'(-x)$ for $0 \leq x \leq \pi$, i.e., $v'(-x) = v'(x)$ for $0 \leq x \leq \pi$. Since v' is 2π-periodic, we conclude that v' is an even function on $\mathbb{R}$. We may assume $v(0) = 0$ (if not, replace v by $v(s) - v(0)$ and replace w by $w(s) + v(0)$). Then

$$v(-x) = \int_0^{-x} v'(s)ds = -\int_0^x v'(-s)ds = -\int_0^x v'(s)ds = -v(x)(\forall\, x \in \mathbb{R}),$$

so v is an odd function on $\mathbb{R}$; moreover $w = v$ since $w(t) = -v(-t)$. Finally, the IC $u(x, 0) = f(x)$ $(0 \leq x \leq \pi)$ gives $f(x) = v(x) + w(x) = 2v(x)$, i.e., $v(x) = \frac{1}{2}f(x)$ for $0 \leq x \leq \pi$. This completes the determination of v: it is the 2π-periodic, odd function on $\mathbb{R}$ which agrees with $\frac{1}{2}f$ on $[0, \pi]$. So d'Alembert's solution can be summarized as follows: define $\widetilde{f}(x) = f(x)$ for $0 \leq x \leq \pi$, $\widetilde{f}(x) = -f(-x)$ for $-\pi \leq x \leq 0$ (the odd extension of f from $[0, \pi]$ to $[-\pi, \pi]$), and then extend $\widetilde{f}$ to be 2π-periodic on $\mathbb{R}$. [Note: if $f(0) = f(\pi) = 0$ and $f \in C^1[0, \pi]$, then $\widetilde{f} \in C^1(\mathbb{R})$; if in addition $f \in C^2[0, \pi]$ and $f''(0) = f''(\pi) = 0$, then $\widetilde{f} \in C^2(\mathbb{R})$.] We obtain d'Alembert's formula for the solution of this IBVP:

$$u(x, t) = \frac{1}{2}\left(\widetilde{f}(x + t) + \widetilde{f}(x - t)\right)$$

(remember, this is the special case where $u_t(x, 0) = 0$ $(0 \leq x \leq \pi)$).

Solution ② (Fourier series). We use separation of variables. We want to find simple harmonics of the string, that is, solutions of the form

$$u(x, t) = v(x)w(t)$$

(often called *fundamental modes*). The v and w here are not the same v and w as above. Using $'$ to mean $\frac{d}{dx}$ for v, and also $\frac{d}{dt}$ for w, the DE $u_{tt} = u_{xx}$ becomes $v(x)w''(t) = v''(x)w(t)$, or (wherever $v(x)w(t) \neq 0$)

$$\frac{w''(t)}{w(t)} = \frac{v''(x)}{v(x)}.$$

The LHS is independent of x and the RHS is independent of t, so both sides are equal to a constant; call it $-\lambda$.

We end up with ODEs for v and w:

$$\begin{array}{rclll} v''(x) + \lambda v(x) & = & 0 & (0 \leq x \leq \pi) & \text{“spatial ODE”} \\ w''(t) + \lambda w(t) & = & 0 & (t \geq 0) & \text{“temporal ODE”} \end{array}$$

Applying the BC to the “spatial ODE”, we get $v(0) = v(\pi) = 0$, leading to the following “eigenvalue problem”: determine for which (in this case real) values of λ there exists a non-trivial (i.e., not $\equiv 0$) solution $v(x)$ of the boundary-value problem (BVP):

$$\begin{array}{lrcll} \text{DE} & v'' + \lambda v & = & 0 & 0 \leq x \leq \pi \\ \text{BC} & v(0) = v(\pi) & = & 0. & \end{array}$$

Case (i) $\lambda < 0$. The general solution of $v'' + \lambda v = 0$ is $c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x)$. $v(0) = 0 \Rightarrow c_1 = 0$, and then $v(\pi) = 0 \Rightarrow c_2 = 0$. No nontrivial solutions.

Case (ii) $\lambda = 0$. The general solution of $v'' = 0$ is $v(x) = c_1 + c_2 x$. $v(0) = 0 \Rightarrow c_1 = 0$, and then $v(\pi) = 0 \Rightarrow c_2 = 0$. No nontrivial solutions.

Case (iii) $\lambda > 0$. The general solution of $v'' + \lambda v = 0$ is $v(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$ $v(0) = 0 \Rightarrow c_1 = 0$. Then $v(\pi) = 0$ (and $c_2 \neq 0$ so v is nontrivial) $\Rightarrow \sin(\sqrt{\lambda}\pi) = 0 \Rightarrow \sqrt{\lambda} \in \{1, 2, 3, \ldots\} \Rightarrow \lambda = n^2$ for $n \in \{1, 2, 3, \ldots\}$. These are the "eigenvalues" of this eigenvalue problem. The corresponding "eigenfunctions" are $\sin(\sqrt{\lambda}x) = \sin(nx)$.

Applying the homogeneous IC $u_t(x, 0) = 0$ to the "temporal ODE," we get $w'(0) = 0$. For $\lambda = n^2$, the general solution of $w'' + \lambda w = 0$ is $c_1 \cos nt + c_2 \sin nt$. The IC $w'(0) = 0$ implies $c_2 = 0$, so $w(t) = c_1 \cos nt$. Thus the *fundamental modes* for this problem are

$$u_n(x, t) = \cos(nt) \sin(nx) \qquad n \in \{1, 2, 3, \ldots\}.$$

Linear combinations of these are also solutions of the DE, the BC, and the one IC $u_t(x, 0) = 0$. To satisfy the IC $u(x, 0) = f(x)$ for $0 \leq x \leq \pi$, we represent $f(x)$ in a Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx).$$

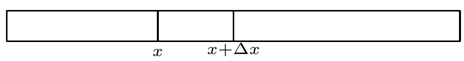
Then (provided this series converges appropriately),

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos(nt) \sin(nx)$$

satisfies the DE, the BC, and both IC. (See Problem 3 on Problem Set 7 for details).

Heat Flow

Consider heat flow in a thin rod with insulated lateral surface.



Assume that the temperature $u(x, t)$ is a function only of horizontal position x and time t. By Newton's law of cooling, the amount of heat flowing from left to right across the point x in time Δt is $-\kappa \frac{\partial u}{\partial x}(x, t)\Delta t$ (proportional to the gradient of temperature), where the constant of proportionality κ is called the *heat conductivity* of the rod. So the net heat flowing *into* the part of rod between x and $x + \Delta x$ in the time interval from t to $t + \Delta t$ is

$$\kappa \frac{\partial u}{\partial x}(x + \Delta x, t)\Delta t - \kappa \frac{\partial u}{\partial x}(x, t)\Delta t.$$

The net heat flowing *into* this part of the rod in this time interval can also be expressed as

$$\overbrace{\rho \Delta x}^{\text{mass}} \cdot \overbrace{c}^{\text{specific heat}} \cdot \overbrace{\frac{\partial u}{\partial t}\Delta t}^{\approx \Delta u},$$

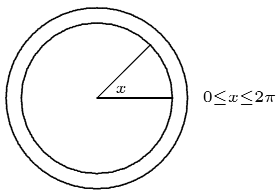
where ρ is the density (mass per unit length) of the rod, and c is the *specific heat* of the rod (the amount of heat needed to raise a unit mass by 1 unit of temperature). Equating these two expressions, dividing by Δt and Δx, and taking the limit as $\Delta x \to 0$, we obtain

$$\kappa u_{xx} = \rho c u_t.$$

Normalizing units so that $\rho c = \kappa$, we obtain the *heat equation* (in one space dimension):

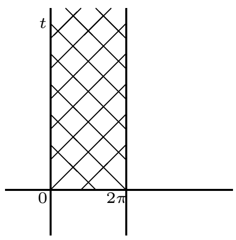
$$u_t = u_{xx}.$$

Fourier considered circular rods of length 2π, leading to the following IBVP with periodic BC:



IBVP:

	DE	$u_t = u_{xx}$	$0 \le x \le 2\pi, t \ge 0$
	IC	$u(x,0) = f(x)$	$0 \le x \le 2\pi$
	periodic BC	$\begin{cases} u(0,t) = u(2\pi,t) \\ u_x(0,t) = u_x(2\pi,t) \end{cases}$	$t \ge 0$



We can view u as defined on $T \times [0,\infty)$ (where $T = S^1$), or as a 2π-periodic function of $x \in \mathbb{R}$ with $t \ge 0$.)

As with the wave equation, we separate variables and look for solutions of the form $u(x,t) = v(x)w(t)$. The DE $u_t = u_{xx}$ becomes $v(x)w'(t) = v''(x)w(t)$, or (wherever $v(x)w(t) \neq 0$)

$$\frac{w'}{w} = \frac{v''}{v};$$

both sides are equal to a constant; call it $-\lambda$. The "spatial ODE" is

$$v''(x) + \lambda v(x) = 0$$

and the "temporal ODE" is

$$w'(t) + \lambda w(t) = 0, \quad (t \ge 0).$$

In this case our eigenvalue problem has periodic boundary conditions:

$$v'' + \lambda v = 0, \qquad (0 \le x \le 2\pi)$$

$$v(0) = v(2\pi), \qquad v'(0) = v'(2\pi).$$

Case (i). $\lambda < 0$. The only solution is $v \equiv 0$.
Case (ii). $\lambda = 0$. There is one linearly independent solution: $v \equiv 1$.
Case (iii). $\lambda > 0$ We must have $\lambda = n^2$ for $n \in \{1, 2, 3, \ldots\}$, now with two linearly independent solutions: $\cos(nt)$ and $\sin(nt)$ (see Problem 1 on Problem Set 7 for details). For $\lambda = n^2$ (with $n \in \{0, 1, 2, \ldots\}$), there is one linearly independent solution of $w' + \lambda w = 0$: $w = e^{-\lambda t}$. Thus the *fundamental modes* for this problem are:

$$u \equiv 1$$

and for $n \in \{1, 2, 3, \ldots\}$:

$$u(x,t) = e^{-n^2 t} \cos nt, \qquad \text{and} \qquad u(x,t) = e^{-n^2 t} \sin nt.$$

To satisfy the IC $u(x, 0) = f(x)$ for $0 \le x \le 2\pi$, we represent $f(x)$ in a Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Then (provided this series converges appropriately)

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-n^2 t} (a_n \cos nx + b_n \sin nx)$$

satisfies the DE, the periodic BC, and the IC.

Remark. This form of the Fourier series of f (viewed as its 2π-periodic extension) is equivalent to the complex exponential form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

For $n \ge 1$,

$$\cos nx = \tfrac{1}{2}\left(e^{inx} + e^{-inx}\right) \qquad \text{and} \qquad \sin nx = \tfrac{1}{2i}\left(e^{inx} - e^{-inx}\right)$$

span the same two-dimensional subspace (over $\mathbb{C}$) as

$$e^{inx} = \cos nx + i \sin nx \qquad \text{and} \qquad e^{-inx} = \cos nx - i \sin nx.$$

The coefficients are related as follows: $c_0 = a_0$, and for $n \ge 1$,

$$c_n = \tfrac{1}{2}(a_n - i b_n), \qquad c_{-n} = \tfrac{1}{2}(a_n + i b_n),$$

$$a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n}).$$

In the inner product

$$\langle g, f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{g(x)} f(x) dx$$

on $L^2(S^1)$, the set

$$\{1, \sqrt{2}\cos nx, \sqrt{2}\sin nx : n \geq 1\}$$

is a complete orthonormal set, giving us the following formulae for a_n and b_n:

$$a_0 = \langle 1, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx,$$

and for $n \geq 1$,

$$a_n = 2\langle \cos nx, f\rangle = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx,$$

$$b_n = 2\langle \sin nx, f\rangle = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx.$$

Caution. Many books will write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

in which case

$$a_0 = 2\langle 1, f\rangle = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(0x)dx.$$

The solution $u(x,t)$ expressed in terms of complex exponentials is

$$u(x,t) = \sum_{\xi\in\mathbb{Z}} \widehat{f}(\xi) e^{-\xi^2 t} e^{i\xi x}$$

where $\widehat{f}(\xi) = \langle e^{ix\xi}, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ix\xi}dx$. Note that if $f \in C^1(T)$ (or even f is continuous and piecewise C^1 on T, meaning f' has only a finite number of jump discontinuities), then $\widehat{f} \in l^1(\mathbb{Z})$. Thus this series for $u(x,t)$ converges absolutely and uniformly for $x \in T$ and $t \geq 0$, and $u(x,0) = f(x)$; moreover, for $t > 0$, this is a C^∞ solution of $u_t = u_{xx}$. This is a consequence of the rapid decay of $e^{-\xi^2 t}$ as $|\xi| \to \infty$ for $t > 0$. But for $t < 0$, we do not expect this series to converge unless $|\widehat{f}(\xi)| \to 0$ extremely fast as $|\xi| \to \infty$. These properties are common for *parabolic equations*: the solution is smooth for $t > 0$, but we *cannot* go backwards in time.

Remark. As for the wave equation, we can also solve IBVP of the form

DE	$u_t = u_{xx}$	$(0 \leq x \leq \pi, t \geq 0)$		
IC	$u(x,0) = f(x)$	$(0 \leq x \leq \pi)$		
BC	$u(0,t) = 0,$	$u(\pi,t) = 0$	$(t \geq 0)$	

(or with BC $u_x(0,t) = 0$, $u_x(\pi,t) = 0$, etc.)