## Lyapunov Stability

The stability of solutions to ODEs was first put on a sound mathematical footing by Lyapunov circa 1890. This theory still dominates modern notions of stability, and provides the foundation upon which alternative notions of stability continue to be built. In this section, we provide a brief introduction to a few of the basic ideas and results in the context of linear homogeneous systems:

$$\text{(LH)} \qquad x' = A(t)x,$$

where $A : \mathbb{R} \to \mathbb{F}^{n\times n}$. We begin by providing a precise definition for these ideas.

A point $x_e \in \mathbb{F}^n$ is said to be an *equilibrium point* for (LH) if the (IVP)

$$x' = A(t)x, \quad x(t_0) = x_e$$

has the unique solution

$$x(t) \equiv x_e \quad \forall\ t \geq t_0.$$

For example, if we take

$$A(t) \equiv \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

then $x_e = (1,\ 0)^T$ is an equilibrium point.

**Definition.**[Lyapunov Stability] The system (LH) is said to be *stable* about the equilibrium point $x_e$ if

$$\forall\ \epsilon > 0\ \exists\ \delta > 0 \text{ such that if } |x(t_0) - x_e| < \delta,\ \text{ then } |x(t) - x_e| < \epsilon \ \ \forall\ t \geq t_0.$$

The system (LH) is said to be *asymptotically stable* about the equilibrium point $x_e$ if

$$\exists\ \delta > 0 \text{ such that if } |x(t_0) - x_e| < \delta,\ \text{ then } |x(t) - x_e| \xrightarrow{t\uparrow\infty} 0.$$

It is possible for a system to be stable but not asymptotically stable.

**Example.**[Stable but not asymptotically stable] Set

$$A(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and consider the equilibrium point $x_e = (0,0)^T$. Since the eigenvalues of $A$ are $\lambda = \pm\mathrm{i}$, the solution to the IVP with $x(t_0) = (\delta_1, \delta_2)^T$ is

$$x(t) = \begin{pmatrix} \delta_1 \cos(t-t_0) + \delta_2 \sin(t-t_0) \\ -\delta_1 \sin(t-t_0) + \delta_2 \cos(t-t_0) \end{pmatrix}.$$

Therefore,

$$|x(t) - x_e| = |x(t_0)| \quad \forall\ t \geq t_0,$$

and so the system is stable, but not asymptotically stable.

With no loss in generality, we need only consider the equilibrium point $x_e = 0$. To see this, let $x_e$ be any equilibrium point for (LH) and let $\Phi(t, t_0)$ be a fundamental matrix for (LH) normalized at $t_0$, e.g. for the (CLH) system $\Phi(t, t_0) = \exp(A(t - t_0))$. Let $x(t)$ be a solution to (LH) and consider the function

$$y(t) := x(t) - x_e = x(t) - \Phi(t, t_0)x_e.$$

Then,

$$\begin{aligned} y'(t) &= \frac{d}{dt}[x(t) - \Phi(t, t_0)x_e] \\ &= x'(t) - \Phi'(t, t_0)x_e \\ &= A(t)x(t) - A(t)\Phi(t, t_0)x_e \\ &= A(t)y(t), \end{aligned}$$

and $y_e = 0$ is an equilibrium point for $y$. We now have the following elementary observations.

**Theorem.**[Fundamental Matrix Characterizations of Stability]

(1) $x_e = 0$ is a stable equilibrium point for (LH) at initial time $t_0$ if and only if there exists $C > 0$ such that $|\Phi(t, t_0)| \le C$ for all $t \ge t_0$.

(2) $x_e = 0$ is an asymptotically stable equilibrium point for (LH) at initial time $t_0$ if and only if $|\Phi(t, t_0)| \to 0$ as $t \to \infty$.

**Proof.** Since $|x(t)| \le |\Phi(t, t_0)|\, |x(t_0)|$, we need only show the forward implication in each the statements of the theorem.
(1) Assume the result is false. Then $x_e = 0$ is stable, but there exists $t_0 < t_k \to \infty$ such that $|\Phi(t_k, t_0)| \uparrow \infty$. In particular, there is a component function $\Phi_{(i,j)}(t, t_0)$ of $\Phi$ such that $|\Phi_{(i,j)}(t_k, t_0)| \to \infty$. Now consider the solution $x(t)$ to the IVP with $x(t_0) = \epsilon \mathsf{e}_j$ for $\epsilon > 0$. We have

$$|x(t_k)| = |\Phi(t_k, t_0)x(t_0)| = \epsilon|\Phi(t_k, t_0)\mathsf{e}_j| \ge \epsilon|\Phi_{(i,j)}(t_k, t_0)| \to \infty,$$

which contradicts the stability of $x_e = 0$.
(2) The proof is essentially the same. Again assume the result is false so that there exists $\epsilon > 0$ and $t_0 < t_k \to \infty$ such that $\inf_k |\Phi(t_k, t_0)| \ge \epsilon$. Then, by passing to a further subsequence if necessary, there is a component function $\Phi_{(i,j)}(t, t_0)$ of $\Phi$ such that $\inf_k |\Phi_{(i,j)}(t_k, t_0)| \ge \epsilon'$ for some $\epsilon' > 0$. Again consider the solution $x(t)$ to the IVP with $x(t_0) = \delta \mathsf{e}_j$ for $\delta > 0$. We have

$$|x(t_k)| = |\Phi(t_k, t_0)x(t_0)| = \delta|\Phi(t_k, t_0)\mathsf{e}_j| \ge \delta|\Phi_{(i,j)}(t_k, t_0)| \ge \delta\epsilon' > 0,$$

which contradicts the asymptotic stability of $x_e = 0$. □

We now consider the stability of solutions to constant coefficient linear homogeneous systems;

$$\text{(CLH)} \qquad x' = Ax,$$

where $A \in \mathbb{F}^{n\times n}$. Let $A \in \mathbb{F}^{n\times n}$ have distinct eigenvalues $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ with multiplicities $n_j$, respectively, and spectral decomposition

$$A = \sum_{j=1}^{m} [\lambda_j P_j + N_j],$$

where

$$P_k P_j = \delta_{kj} P_j, \quad P_k N_j = N_j P_k = \delta_{kj} N_j$$
$$P_j^2 = P_j, \quad N_j^{n_j} = 0, \quad \text{and} \quad \sum_{s=1}^{m} P_s = I,$$

for all $1 \le j, k \le m$ with

$$\delta_{kj} = \begin{cases} 1 & k = j, \\ 0 & k \ne j. \end{cases}$$

From our previous work on (CLH), we know that

$$\Phi(t, t_0) = e^{A(t-t_0)} = \sum_{j=1}^{m} e^{\lambda_j (t-t_0)} \left[ P_j + \sum_{k=1}^{n_j - 1} \frac{(t-t_0)^k}{k!} N_j^k \right].$$

The following stability theorem follows from this representation.

**Theorem.**[Stability for (CLH)] Let $A \in \mathbb{F}^{n\times n}$ have the spectral decomposition described above.

(1) (CLH) is stable about $x_e = 0$ if and only if $\mathcal{R}e(\lambda) \le 0$ for all $\lambda \in \sigma(A)$, and for all $\lambda_j \in \sigma(A)$ for which $\mathcal{R}e(\lambda_j) = 0$ it must be the case that $N_j = 0$, i.e., $\lambda_j$ is semi-simple.

(2) (CLH) is asymptotically stable about $x_e = 0$ if and only if $\mathcal{R}e(\lambda) < 0$ for all $\lambda \in \sigma(A)$.

**Example.** A broad result of this type does not extend to non-constant coefficient (LH) systems even if the eigenvalues of $A(t)$ are constant with negative del parts for all $t$. For example, consider the matrix function

$$A(t) = \begin{bmatrix} -4 & 3e^{-8t} \\ -e^{-8t} & 0 \end{bmatrix}.$$

It is easily seen that the eigenvalues for $A(t)$ are $\lambda = -1, -3$ for all $t \in \mathbb{R}$. On the other hand, a solution to the IVP $(x' = A(t)x,\ x(0) = \epsilon(1,1)^T)$ is given by

$$x(t) = \epsilon \begin{pmatrix} 3e^{-5t} - 2e^{-7t} \\ 2e^{t} - e^{3t} \end{pmatrix},$$

with $x(t) \to (0, -\infty)$ as $t \to \infty$.

**Lyapunov Functions**

A function $V : \mathbb{F}^n \to \mathbb{R}$ is said to be a *positive definite function* if

1. $V(x) \geq 0 \quad \forall\, x \in \mathbb{R}^n$,
2. $V(x) = 0$ if and only if $x = 0$, and
3. for all $\alpha \in \mathbb{R}$ the set $\{x \in \mathbb{F}^n \,:\, V(x) \leq \alpha\}$ is compact.

Given $f \in C^1[[t_0, \infty), \mathbb{F}^n]$, a *Lyapunov function* for the differential equation

$$\text{DE} \qquad x' = f(t, x)$$

is any continuously differentiable *positive definite function* $V : \mathbb{F}^n \to \mathbb{R}$ such that for every solution $x$ to DE on $I = [t_0, \infty)$, we have

$$\frac{d}{dt} V(x(t)) = \mathcal{R}e(\nabla V(x(t))^H x'(t)) = \mathcal{R}e(\langle \nabla V(x(t)),\, x'(t)\rangle) \leq 0.$$

The following stability result is derived from the existence of a Lyapunov function.

**Theorem.** If DE (with $f \in C^1$) has a Lyapunov function, then all solutions to DE are bounded, i.e., if $x(t)$ is a solution to DE on $I$, then there is an $R \geq 0$ such that $|x(t)| \leq R$ for all $t \in I$.

**Proof.** Let $x(t)$ be a solution to DE on $I$. Then

$$V(x(t)) = V(x(t_0)) + \int_{t_0}^{t} \frac{d}{dt} V(x(t)) dt \leq V(x(t_0)).$$

The result follows since the set $\{x \in \mathbb{F}^n \,:\, V(x) \leq V(x(t_0))\}$ is compact. $\square$

We also have the following result on asymptotic stability.

**Theorem.** Suppose that the autonomous DE $x' = f(x)$ (with $f \in C^1$) has a Lyapunov function $V$ for which

$$\mathcal{R}e(\nabla V(x)^H f(x)) < 0 \quad \text{whenever } x \neq 0 \text{ with } \mathcal{R}e(\nabla V(x)^H f(x)) = 0 \text{ when } x = 0.$$

Then every solution to DE is asymptotically stable to zero.

**Proof.** Suppose to the contrary that there is a solution $x(t)$ to DE on I that is not asymptotically stable to zero. Then $u(t) := V(x(t))$ is decreasing and non-negative. Hence, there is an $\epsilon > 0$ such that $u(t) \downarrow \epsilon$. That is, for all $t \in I$, $x(t)$ resides in the compact set

$$C := \{x \in \mathbb{F}^n \,:\, 0 < \epsilon \leq V(x) \leq V(x(t_0))\}.$$

Since $0 \notin C$, $C$ is compact, and $z \mapsto \mathcal{R}e(\nabla V(z)^H f(z))$ is continuous, there exists $a > 0$ such that

$$\sup_{z \in C} \mathcal{R}e(\nabla V(z)^H f(z)) \leq -a.$$

But then

$$V(x(t)) = V(x(t_0)) + \int_{t_0}^{t} \frac{d}{dt} V(x(t)) dt \leq V(x(t_0)) - a(t - t_0) \quad \forall\, t \geq t_0.$$

This contradicts the fact that $V(x) \geq 0$ for all $x$. Hence $x(t)$ must be asymptotic to zero.

**A Lyapunov Function for Autonomous Linear Homogeneous Systems**

In general, building a Lyapunov function in a specific instance is far from a straight forward task. We walk through the process in the case of an autonomous linear homogeneous systems:

$$\text{(CLH)}: \qquad x' = Ax \qquad \text{where } A \in \mathbb{F}^{n\times n}.$$

We begin by developing some auxiliary variational properties of the *spectral abscissa* mapping.

**Definition.**[The Spectral Abscissa] The spectral abscissa is the mapping $\alpha : \mathbb{F}^{n\times n} \to \mathbb{R}$ given by

$$\alpha(A) := \max\{\mathcal{R}e(\lambda) : \det(A - \lambda I) = 0\},$$

that is, it the maximum of the real part of the spectrum of $A$.

Our next result shows that the spectral abscissa of a matrix can be approximated to any degree of accuracy by the eigenvalues of a Hermitian matrix.

**Definition.**

(1) The *general linear group* on $\mathbb{F}^n$ is defined to be the set

$$GL(n,\mathbb{F}) := \left\{A \in \mathbb{F}^{n\times n} \;:\; \det(A) \neq 0\right\}.$$

(2) $\mathcal{H}^n$ is the subspace of $\mathbb{C}^{n\times n}$ of Hermitian matrices, and $\mathcal{H}^n_+ \subset \mathcal{H}^n$ is the cone of positive semidefinite Hermitian matrices whose interior relative to $\mathcal{H}^n$ is the set of positive definite Hermitian matrices $\mathcal{H}^n_{++}$. The cone $\mathcal{H}^n_+$ defines a partial ordering "$\preceq$" on $\mathcal{H}^n$ by

$$A \preceq B \iff B - A \in \mathcal{H}^n_+.$$

(3) $\mathcal{S}^n$ is the subspace of $\mathbb{R}^{n\times n}$ of symmetric matrices, and $\mathcal{S}^n_+ \subset \mathcal{S}^n$ is the cone of positive semidefinite symmetric matrices whose interior relative to $\mathcal{S}^n$ is the set of positive definite symmetric matrices $\mathcal{S}^n_{++}$. The cone $\mathcal{S}^n_+$ defines a partial ordering "$\preceq$" on $\mathcal{S}^n$ by

$$A \preceq B \iff B - A \in \mathcal{S}^n_+.$$

**Lemma.** The mapping $\mathcal{H} : \mathbb{C}^{n\times n} \to \mathcal{H}^n$ given by $\mathcal{H}(M) = \frac{1}{2}(M + M^*)$ is the orthogonal projection of $\mathbb{C}^{n\times n}$ onto the subspace $\mathcal{H}^n$ in the Frobenius inner product. Similarly, the mapping $\mathcal{S} : \mathbb{R}^{n\times n} \to \mathcal{S}^n$ given by $\mathcal{S}(M) = \frac{1}{2}(M + M^T)$ is the orthogonal projection of $\mathbb{R}^{n\times n}$ onto the subspace $\mathcal{S}^n$.

**Lemma.**[Spectral Abscissa Approximation I] Given $A \in \mathbb{C}^{n\times n}$, we have

$$\alpha(A) = \inf\left\{\gamma \;:\; X \in GL(n,\mathbb{C}^n) \text{ and } \mathcal{H}(XAX^{-1}) \preceq \gamma I_n\right\}. \tag{1}$$

*Remark.* The same result holds over real matrices, i.e. given $A \in \mathbb{R}^{n\times n}$, we have

$$\alpha(A) = \inf\left\{\gamma \,:\, X \in GL(n,\mathbb{R}^n) \text{ and } \mathcal{S}(XAX^{-1}) \preceq \gamma I_n\right\}. \tag{2}$$

*Remark.* Note that if $M \in \mathcal{H}^n$, then $M \preceq \gamma I$ if and only if $\lambda \le \gamma$ for all $\lambda \in \sigma(M)$. Indeed, since $M$ is unitarily diagonalizable, we have $\sigma(\gamma I - M) = \{\gamma - \lambda \,:\, \lambda \in \sigma(M)\}$.

**Proof.** First suppose that $X \in GL(n,\mathbb{C}^n)$ is such that $\mathcal{H}(XAX^{-1}) \preceq \gamma I_n$. Then, for every $u \in \mathbb{C}^n$ with $|u| = 1$, we have

$$\begin{aligned} 2\gamma = 2\gamma|u|^2 &\ge u^H XAX^{-1}u + u^H X^{-H}A^H X^H u \\ &= u^H XAX^{-1}u + \overline{u^H XAX^{-1}u} = 2\mathcal{R}e(u^H XAX^{-1}u). \end{aligned}$$

Let $\lambda \in \sigma(A) = \sigma(XAX^{-1})$, and let $u$ be a unit eigenvector for $XAX^{-1}$ associated with the eigenvalue $\lambda$. Then the previous inequality tells us that

$$\gamma \ge \mathcal{R}e(u^H XAX^{-1}u) = \mathcal{R}e(\lambda).$$

Hence, the infimum on the right-hand side of (1) exceeds $\alpha(A)$.

Next we show that for every $\gamma > \alpha(A)$ there exists $X \in GL(n,\mathbb{C})$ such that $\mathcal{H}(XAX^{-1}) \le \gamma I$ which establishes the result. To this end, let $A = Q^H TQ$ be the Shur form of $A$, i.e., $Q$ is unitary and $T$ is upper triangular:

$$T = \begin{bmatrix} \lambda_1 & t_{12} & t_{13} & \dots & t_{1n} \\ 0 & \lambda_2 & t_{23} & \dots & t_{2n} \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & & \lambda_n \end{bmatrix}.$$

Let $\delta > 0$ and set $D_\delta = \operatorname{diag}(\delta, \delta^2, \delta^3, \dots, \delta^n)$. Then

$$D_\delta^{-1}TD_\delta = \begin{bmatrix} \lambda_1 & \delta t_{12} & \delta^2 t_{13} & \dots & \delta^{n-1}t_{1n} \\ 0 & \lambda_2 & \delta t_{23} & \dots & \delta^{n-2}t_{2n} \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & & \delta t_{(n-1)n} \\ 0 & \dots & & & \lambda_n \end{bmatrix}.$$

Set $X_\delta = D_\delta^{-1}Q$ so that $X_\delta^{-1} = Q^H D_\delta$ and $X_\delta AX_\delta^{-1} = D_\delta^{-1}TD_\delta$. Hence

$$\mathcal{H}(X_\delta AX_\delta^{-1}) = \begin{bmatrix} \mathcal{R}e(\lambda_1) & \frac{\delta}{2}t_{12} & \frac{\delta^2}{2}t_{13} & \dots & \frac{\delta^{n-1}}{2}t_{1n} \\ \frac{\delta}{2}\bar{t}_{12} & \mathcal{R}e(\lambda_2) & \frac{\delta}{2}t_{23} & \dots & \frac{\delta^{n-2}}{2}t_{2n} \\ \vdots & & \ddots & & \vdots \\ \frac{\delta^{n-2}}{2}\bar{t}_{1(n-1)} & \dots & & & \frac{\delta}{2}t_{(n-1)n} \\ \frac{\delta^{n-1}}{2}\bar{t}_{1n} & \dots & & & \mathcal{R}e(\lambda_n) \end{bmatrix} \to \operatorname{diag}(\mathcal{R}e(\lambda_1), \dots, \mathcal{R}e(\lambda_n)).$$

Therefore, $\sigma(\mathcal{H}(X_\delta A X_\delta^{-1})) \to \{\mathcal{R}e(\lambda_1), \ldots, \mathcal{R}e(\lambda_n)\}$. Consequently, if $\alpha(A) < \gamma$, there exists $\bar{\delta} > 0$ such that $\alpha(\mathcal{H}(X_\delta A X_\delta^{-1})) < \gamma$ for all $\delta \in (0, \bar{\delta})$, or equivalently, $\mathcal{H}(X_\delta A X_\delta^{-1}) \prec \gamma I$. $\square$

Next observe that $M \in \mathcal{H}^n$ satisfies

$$\exists\, \gamma \in \mathbb{R} \text{ s.t. } M \preceq \gamma I \iff W^H M W \preceq \gamma W^H W \quad \forall\, W \in \mathbb{F}^{n\times n}.$$

Here, the implication "$\Leftarrow$" is obvious since we can take $W = I$. The reverse implication follows since $M \preceq \gamma I$ if and only if $(\gamma I - M)$ is positive semi-definite which implies that $W^H(\gamma I - M)W$ is also Hermitian positive semidefinite for all $W \in \mathbb{C}^{n\times n}$. Therefore, we have

$$\begin{gathered} \mathcal{H}(XAX^{-1}) \preceq \gamma I \\ \Longleftrightarrow \\ X^H \mathcal{H}(XAX^{-1}) X \preceq \gamma X^H X \\ \Longleftrightarrow \\ X^H X A + A^H X^H X \preceq 2\gamma X^H X \\ \Longleftrightarrow \\ PA + A^H P \preceq 2\gamma P \qquad \text{with } 0 \prec P = X^H X. \end{gathered}$$

By combining this observation with the preceding lemma, we obtain the following result.

**Theorem.**[Spectral Abscissa Approximation II] Given $A \in \mathbb{C}^{n\times n}$, we have

$$\alpha(A) = \inf\left\{\gamma \,:\, 0 \prec P \in \mathcal{H}^n \text{ and } PA + A^H P \preceq 2\gamma P\right\}. \tag{3}$$

Suppose that $\alpha(A) < 0$. This theorem tells us that to every $\gamma \in (\alpha(A), 0)$ there is a positive definite matrix $P_g$ satisfying

$$P_\gamma A + A^H P_\gamma \preceq 2\gamma P_\gamma.$$

It is now straightforward to show that the function $V(x) = x^H P_\gamma x$ is a Lyapunov function for (CLH). Moreover we have the following theorem.

**Theorem.**[Exponential Stability of (CLH)] Let $A \in \mathbb{C}^{n\times n}$ be such that $\alpha(A) < 0$. Then, given $\alpha(A) < \gamma < 0$, there exists $0 \prec P \in \mathcal{H}^n$ such that

$$PA + A^H P \preceq 2\gamma P.$$

Moreover, for every solution $x(t)$ to (CLH),

$$|x(t)| \le \sqrt{\kappa(P)}\,|x(t_0)| e^{\gamma(t-t_0)} \qquad \forall\, t \ge t_0,$$

where $\kappa(P)$ is the condition number for the matrix $P$.

*Remark.* If $A \in \mathbb{R}^{n\times n}$, is such that $\alpha(A) < 0$, then, given $\alpha(A) < \gamma < 0$, there exists $0 \prec P \in \mathcal{S}^n$ such that

$$PA + A^H P \preceq 2\gamma P.$$

Moreover, for every solution $x(t)$ to (CLH),

$$|x(t)| \le \sqrt{\kappa(P)}\,|x(t_0)| e^{\gamma(t-t_0)} \qquad \forall\, t \ge t_0,$$

where $\kappa(P)$ is the condition number for the matrix $P$.