Linear ODE

Let $I \subset \mathbb{R}$ be an interval (open or closed, finite or infinite — at either end). Suppose $A : I \to \mathbb{F}^{n\times n}$ and $b : I \to \mathbb{F}^n$ are continuous. The DE

$$(*) \qquad x' = A(t)x + b(t)$$

is called a first-order linear [system of] ODE[s] on I. Since $f(t,x) \equiv A(t)x+b(t)$ is continuous in t, x on $I \times \mathbb{F}^n$ and, for any compact subinterval $[c,d] \subset I$, f is uniformly Lipschitz in x on $[c,d]\times\mathbb{F}^n$ (with Lipschitz constant $\max_{c\le t\le d}|A(t)|$), we have global existence and uniqueness of solutions of the IVP

$$x' = A(t)x + b(t), \qquad x(t_0) = x_0$$

on all of I (where $t_0 \in I$, $x_0 \in \mathbb{F}^n$).

If $b \equiv 0$ on I, $(*)$ is called a linear homogeneous system (LH).

If $b \not\equiv 0$ on I, $(*)$ is called a linear inhomogeneous system (LI).

Fundamental Theorem for LH. *The set of all solutions of (LH) $x' = A(t)x$ on I forms an n-dimensional vector space over $\mathbb{F}$ (in fact, a subspace of $C^1(I,\mathbb{F}^n)$).*

Proof. Clearly $x_1' = Ax_1$ and $x_2' = Ax_2$ imply $(c_1x_1 + c_2x_2)' = A(c_1x_1 + c_2x_2)$, so the set of solutions of (LH) forms a vector space over $\mathbb{F}$, which is clearly a subspace of $C^1(I,\mathbb{F}^n)$. Fix $\tau \in I$, and let $y_1,\ldots,y_n$ be a basis for $\mathbb{F}^n$. For $1 \le j \le n$, let $x_j(t)$ be the solution of the IVP $x' = Ax$, $x(\tau) = y_j$. Then $x_1(t),\ldots,x_n(t)$ are linearly independent in $C^1(I,\mathbb{F}^n)$; indeed,

$$\sum_{j=1}^n c_jx_j(t) = 0 \quad \text{in} \quad C^1(I,\mathbb{F}^n)$$

$$\Rightarrow$$

$$\sum_{j=1}^n c_jx_j(t) = 0 \quad \forall\, t \in I$$

$$\Rightarrow$$

$$\sum_{j=1}^n c_jy_j = \sum c_jx_j(\tau) = 0$$

$$\Rightarrow$$

$$c_j = 0 \quad j = 1,2,\ldots,n.$$

Now if $x(t)$ is any solution of (LH), there exist unique $c_1,\ldots,c_n$ such that $x(\tau) = c_1y_1 + \cdots + c_ny_n$. Clearly $c_1x_1(t) + \cdots + c_nx_n(t)$ is *a* solution of the IVP

$$x' = A(t)x, \quad x(\tau) = c_1y_1 + \cdots c_ny_n,$$

so by uniqueness, $x(t) = c_1x_1(t) + \cdots + c_nx_n(t)$ for all $t \in I$. Thus $x_1(t),\ldots,x_n(t)$ span the vector space of all solutions of (LH) on I. So they form a basis, and the dimension is n. □

Remark. Define the linear operator $L : C^1(I, \mathbb{F}^n) \to C^0(I, \mathbb{F}^n)$ by $Lx = \left(\frac{d}{dt} - A(t)\right) x$, i.e., $(Lx)(t) = x'(t) - A(t)x(t)$ for $x(t) \in C^1(I, \mathbb{F}^n)$. L is called a *linear differential operator*. The solution space in the previous theorem is precisely the null space of L. Thus the null space of L is finite dimensional and has dimension n.

Definition. A set $\{\varphi_1, \ldots, \varphi_n\}$ of solutions of (LH) $x' = Ax$ on I is said to be a *fundamental set* of solutions if it is a basis for the vector space of all solutions. If $\Phi : I \to \mathbb{F}^{n\times n}$ is an $n \times n$ matrix function of $t \in I$ whose columns form a fundamental set of solutions of (LH), then $\Phi(t)$ is called a *fundamental matrix* for (LH) $x' = A(t)x$. Checking columnwise shows that a fundamental matrix satisfies

$$\Phi'(t) = A(t)\Phi(t).$$

Definition. If $X : I \to \mathbb{F}^{n\times k}$ is in $C^1(I, \mathbb{F}^{n\times k})$, we say that X is an $[n \times k]$ matrix solution of (LH) if $X'(t) = A(t)X(t)$. Clearly $X(t)$ is a matrix solution of (LH) if and only if each column of $X(t)$ is a solution of (LH). (We will mostly be interested in the case $k = n$.)

Theorem. Let $A : I \to \mathbb{F}^{n\times n}$ be continuous, where $I \subset \mathbb{R}$ is an interval, and suppose $X : I \to \mathbb{F}^{n\times n}$ is an $n \times n$ matrix solution of (LH) $x' = A(t)x$ on I, i.e., $X'(t) = A(t)X(t)$ on I. Then $\det(X(t))$ satisfies the linear homogeneous first-order scalar ODE

$$\det(X(t))' = \operatorname{tr}(A(t))\det X(t),$$

and so for all $\tau, t \in I$,

$$\det X(t) = (\det(X(\tau))) \exp \int_\tau^t \operatorname{tr}(A(s))ds.$$

Proof. Let $x_{ij}(t)$ denote the ij^{th} element of $X(t)$, and let $\hat{X}_{ij}(t)$ denote the $(n-1) \times (n-1)$ matrix obtained from $X(t)$ by deleting its ith row and jth column. The co-factor representation of the determinant gives

$$\det(X) = \sum_{j=1}^{n} (-1)^{(i+j)} x_{ij} \det(\hat{X}_{ij}), \quad i = 1, 2, \ldots, n\ .$$

Hence

$$\frac{\partial}{\partial x_{ij}} \det(X) = (-1)^{(i+j)} \det(\hat{X}_{ij}),$$

and so by the chain rule

$$\begin{aligned}(\det X(t))' &= \sum_{j=1}^{n} (-1)^{(1+j)} x'_{1j}(t) \det(\hat{X}_{ij}(t)) + \cdots + \sum_{j=1}^{n} (-1)^{(n+j)} x'_{nj}(t) \det(\hat{X}_{ij}(t)) \\ &= \det \begin{bmatrix} x'_{11} & x'_{12} & \cdots & x'_{1n} \\ & (\text{remaining } x_{ij}) & & \end{bmatrix} + \cdots + \det \begin{bmatrix} & (\text{remaining } x_{ij}) & & \\ x'_{n1} & x'_{n2} & \cdots & x'_{nn} \end{bmatrix}.\end{aligned}$$

Now by (LH)

$$\begin{aligned}[\ x'_{11}\ \ x'_{12}\ \ \cdots\ \ x'_{1n}\ ] &= [\Sigma_k a_{1k}x_{k1}\cdots\Sigma_k a_{1k}x_{kn}] \\ &= a_{11}[x_{11}\cdots x_{1n}] + a_{12}[x_{21}\cdots x_{2n}] + \cdots + a_{1n}[x_{n1}\cdots x_{nn}].\end{aligned}$$

Subtracting $a_{12}[x_{21}\cdots x_{2n}]+\cdots+a_{1n}[x_{n1}\cdots x_{nn}]$ from the first row of the matrix in the first determinant on the RHS doesn't change that determinant. A similar argument applied to the other determinants gives

$$\begin{aligned}(\det X(t))' =&\det\begin{bmatrix} a_{11}[x_{11}\cdots x_{1n}] \\ (\text{remaining } x_{ij})\end{bmatrix} + \cdots + \det\begin{bmatrix}(\text{remaining } x_{ij}) \\ a_{nn}[x_{n1}\cdots x_{nn}]\end{bmatrix} \\ =&(a_{11}+\cdots+a_{nn})\det X(t) = \operatorname{tr}(A(t))\det X(t).\end{aligned}$$

□

Corollary. Let $X(t)$ be an $n\times n$ matrix solution of (LH) $x' = A(t)x$. Then either

$$(\forall\, t\in I)\quad \det X(t)\neq 0 \quad \text{or} \quad (\forall\, t\in I)\quad \det X(t) = 0.$$

Corollary. Let $X(t)$ be an $n\times n$ matrix solution of (LH) $x' = A(t)x$. Then the following statements are equivalent.

(1) $X(t)$ is a fundamental matrix for (LH) on I.

(2) $(\exists\,\tau\in I)$ $\det X(\tau)\neq 0$ (i.e., columns of X are linearly independent at τ)

(3) $(\forall\, t\in I)$ $\det X(t)\neq 0$ (i.e., columns of X are linearly independent at every $t\in I$).

Definition. If $X(t)$ is an $n\times n$ matrix solution of (LH) $x' = A(t)x$, then $\det(X(t))$ is often called the Wronskian [of the columns of $X(t)$].

Remark. This is not quite standard notation for general LH systems $x' = A(t)x$. It is used most commonly when $x' = A(t)x$ is the first-order system equivalent to a scalar n^{th}-order linear homogeneous ODE.

Theorem. Suppose $\Phi(t)$ is a fundamental matrix for (LH) $x' = A(t)x$ on I.

(a) If $c\in\mathbb{F}^n$, then $x(t) = \Phi(t)c$ is a solution of (LH) on I.

(b) If $x(t)\in C^1(I,\mathbb{F}^n)$ is any solution of (LH) on I, then there exists a unique $c\in\mathbb{F}^n$ for which $x(t) = \Phi(t)c$.

Proof. The theorem just restates that the columns of $\Phi(t)$ for a basis for the set of solutions of (LH). $\square$

Remark. The *general solution* of (LH) is $\Phi(t)c$ for arbitrary $c \in \mathbb{F}^n$, where $\Phi(t)$ is a fundamental matrix.

Theorem. Suppose $\Phi(t)$ is a fundamental matrix (F.M.) for (LH) $x' = A(t)x$ on I.

(a) If $C \in \mathbb{F}^{n\times n}$ is invertible, then $X(t) = \Phi(t)C$ is also a F.M. for (LH) on I.

(b) If $X(t) \in C^1(I, \mathbb{F}^{n\times n})$ is any F.M. for (LH) on I, then there exists a unique invertible $C \in \mathbb{F}^{n\times n}$ for which $X(t) = \Phi(t)C$.

Proof. For (a), observe that

$$X'(t) = \Phi'(t)C = A(t)\Phi(t)C = A(t)X(t),$$

so $X(t)$ is a matrix solution, and $\det X(t) = (\det \Phi(t))(\det C) \neq 0$.

For (b), set $\Psi(t) = \Phi(t)^{-1}X(t)$. Then $X = \Phi\Psi$, so

$$\Phi'\Psi + \Phi\Psi' = (\Phi\Psi)' = X' = AX = A\Phi\Psi = \Phi'\Psi,$$

which implies that $\Phi\Psi' = 0$. Since $\Phi(t)$ is invertible for all $t \in I$, $\Psi'(t) \equiv 0$ on I. So $\Psi(t)$ is a constant invertible matrix C. Since $C = \Psi = \Phi^{-1}X$, we have $X(t) = \Phi(t)C$. $\square$

Remark. If $B(t) \in C^1(I, \mathbb{F}^{n\times n})$ is invertible for each $t \in I$, then

$$\frac{d}{dt}(B^{-1}(t)) = -B^{-1}(t)B'(t)B^{-1}(t).$$

The proof is to differentiate $I = BB^{-1}$:

$$0 = \frac{d}{dt}(I) = \frac{d}{dt}(B(t)B^{-1}(t)) = B(t)\frac{d}{dt}(B^{-1}(t)) + B'(t)B^{-1}(t).$$

Adjoint Systems

Let $\Phi(t)$ be a F.M. for (LH) $x' = A(t)x$. Then

$$(\Phi^{-1})' = -\Phi^{-1}\Phi'\Phi^{-1} = -\Phi^{-1}A\Phi\Phi^{-1} = -\Phi^{-1}A.$$

Taking conjugate transposes, $(\Phi^{*-1})' = -A^*\Phi^{*-1}$. So $\Phi^{*-1}(t)$ is a F.M. for the *adjoint system* (LH*) $x' = -A^*(t)x$.

Theorem. If $\Phi(t)$ is a F.M. for (LH) $x' = A(t)x$ and $\Psi(t) \in C^1(I, \mathbb{F}^{n\times n})$, then $\Psi(t)$ is a F.M. for (LH*) $x' = -A^*(t)x$ if and only if $\Psi^*(t)\Phi(t) = C$, where C is a constant invertible matrix.

Proof. Suppose $\Psi(t)$ is a F.M. for (LH*). Since $\Phi^{*-1}(t)$ is also a F.M. for (LH*), $\exists$ an invertible $C \in \mathbb{F}^{n\times n} \ni \Psi(t) = \Phi^{*-1}(t)C^*$, i.e., $\Psi^* = C\Phi^{-1}$, $\Psi^*\Phi = C$. Conversely, if $\Psi^*(t)\Phi(t) = C$ (invertible), then $\Psi^* = C\Phi^{-1}$, $\Psi = \Phi^{*-1}C^*$, so Ψ is a F.M. for (LH*). $\square$

Normalized Fundamental Matrices

Definition. A F.M. $\Phi(t)$ for (LH) $x' = A(t)x$ is called *normalized at time* τ if $\Phi(\tau) = I$, the identity matrix. (Convention: if not stated otherwise, a normalized F.M. usually means normalized at time $\tau = 0$.)

Facts:

(1) For a given τ, the F.M. of (LH) normalized at τ exists and is unique. (*Proof.* The j^{th} column of $\Phi(t)$ is the solution of the IVP $x' = A(t)x$, $x(\tau) = e_j$.)

(2) If $\Phi(t)$ is the F.M. for (LH) normalized at τ, then the solution of the IVP $x' = A(t)x$, $x(\tau) = y$ is $x(t) = \Phi(t)y$. (*Proof.* $x(t) = \Phi(t)y$ satisfies (LH) $x' = A(t)x$, and $x(\tau) = \Phi(\tau)y = Iy = y$.)

(3) For any fixed τ, t, the solution operator S_τ^t for (LH), mapping $x(\tau)$ into $x(t)$, is a *linear* operator on $\mathbb{F}^n$, and its matrix is the F.M. $\Phi(t)$ for (LH) normalized at τ, evaluated at t.

(4) If $\Phi(t)$ is *any* F.M. for (LH), then for fixed τ, $\Phi(t)\Phi^{-1}(\tau)$ is the F.M. for (LH) normalized at τ. (*Proof.* It is a F.M. taking the value I at τ.) Thus:
(a) $\Phi(t)\Phi^{-1}(\tau)$ is the matrix of the solution operator S_τ^t for (LH); and
(b) the solution of the IVP $x' = A(t)x$, $x(\tau) = y$ is $x(t) = \Phi(t)\Phi^{-1}(\tau)y$.

Reduction of Order for (LH) $\mathbf{x}' = \mathbf{A}(\mathbf{t})\mathbf{x}$

If m ($< n$) linearly independent solutions of the $n \times n$ linear homogeneous system $x' = A(t)x$ are known, then one can derive an $(n-m) \times (n-m)$ system for obtaining $n - m$ more linearly independent solutions. See Coddington & Levinson for details.

Inhomogeneous Linear Systems

We now want to express the solution of the IVP

$$x' = A(t)x + b(t), \quad x(t_0) = y$$

for the linear inhomogeneous system

$$\text{(LI)} \qquad x' = A(t)x + b(t)$$

in terms of a F.M. for the associated homogeneous system

$$\text{(LH)} \qquad x' = A(t)x.$$

Variation of Parameters

Let $\Phi(t)$ be any F.M. for (LH). Then for any constant vector $c \in \mathbb{F}^n$, $\Phi(t)c$ is a solution of (LH). We will look for a solution of (LI) of the form

$$x(t) = \Phi(t)c(t)$$

(varying the "constants" — elements of c). Plugging into (LI), we want

$$(\Phi c)' = A\Phi c + b,$$

or equivalently

$$\Phi' c + \Phi c' = A\Phi c + b.$$

Since $\Phi' = A\Phi$, this gives $\Phi c' = b$, or $c' = \Phi^{-1}b$. So let

$$c(t) = c_0 + \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds$$

for some constant vector $c_0 \in \mathbb{F}^n$, and let $x(t) = \Phi(t)c(t)$. These calculations show that $x(t)$ is a solution of (LI). To satisfy the initial condition $x(t_0) = y$, we take $c_0 = \Phi^{-1}(t_0)y$, and obtain

$$x(t) = \Phi(t)\Phi^{-1}(t_0)y + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)b(s)ds.$$

In words, this equation states that

$$\left\{ \begin{array}{c} \text{soln of (LI)} \\ \text{with I.C. } x(t_0) = y \end{array} \right\} = \left\{ \begin{array}{c} \text{soln of (LH)} \\ \text{with I.C. } x(t_0) = y \end{array} \right\} + \left\{ \begin{array}{c} \text{soln of (LI)} \\ \text{with homog. I.C. } x(t_0) = 0 \end{array} \right\}.$$

Viewing y as arbitrary, we find that the *general solution of (LI)* equals the *general solution of (LH)* plus a *particular solution of (LI).*

Recall that $\Phi(t)\Phi^{-1}(t_0)$ is the matrix of $S_{t_0}^t$, and $\Phi(t)\Phi^{-1}(s)$ is the matrix of S_s^t. So the above formula for the solution of the IVP can be written just in terms of the solution operator:

Duhamel's Principle. If S_τ^t is the solution operator for (LH), then the solution of the IVP $x' = A(t)x + b(t)$, $x(t_0) = y$ is

$$x(t) = S_{t_0}^t y + \int_{t_0}^{t} S_s^t(b(s))ds.$$

Remark. So the effect of the inhomogeneous term $b(t)$ in (LI) is the same as adding an additional IC $b(s)$ at each time $s \in [t_0, t]$ and integrating these solutions $S_s^t(b(s))$ of (LH) with respect to $s \in [t_0, t]$.

Constant Coefficient Systems

Consider the linear homogeneous constant-coefficient first-order system

$$\text{(LHC)} \qquad x' = Ax,$$

where $A \in \mathbb{F}^{n\times n}$ is a constant matrix. The F.M. of (LHC), normalized at 0, is $\Phi(t) = e^{tA}$. This is justified as follows. Recall that

$$e^B \equiv \sum_{j=0}^{\infty} \frac{1}{j!} B^j$$

where $B^0 \equiv I$. So $\Phi(0) = I$. Term by term differentiation is justified in the series for e^{tA} :

$$\begin{aligned} \Phi'(t) &= \frac{d}{dt}(e^{tA}) = \sum_{j=0}^{\infty} \frac{1}{j!}\frac{d}{dt}(tA)^j \\ &= \sum_{j=1}^{\infty} \frac{1}{(j-1)!} t^{j-1} A^j = A \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k = Ae^{tA} = A\Phi(t). \end{aligned}$$

We can express e^{tA} using the Jordan form of A: if $P^{-1}AP = J$ is in Jordan form where $P \in \mathbb{F}^{n\times n}$ is invertible (assume $\mathbb{F} = \mathbb{C}$ if A has any nonreal eigenvalues), then $A = PJP^{-1}$, so $e^{tA} = e^{tPJP^{-1}} = Pe^{tJ}P^{-1}$. If

$$J = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_s \end{bmatrix}$$

where each J_k is a single Jordan block, then

$$e^{tJ} = \begin{bmatrix} e^{tJ_1} & & & 0 \\ & e^{tJ_2} & & \\ & & \ddots & \\ 0 & & & e^{tJ_s} \end{bmatrix}.$$

Finally, if

$$J_k = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$

is $l \times l$, then

$$e^{tJ_k} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{l-1}}{(l-1)!} \\ & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t}{2!} \\ & & & & t \\ 0 & & & & 1 \end{bmatrix}.$$

The solution of the inhomogeneous IVP $x' = Ax + b(t)$, $x(t_0) = y$ is

$$x(t) = e^{(t-t_0)A}y + \int_{t_0}^{t} e^{(t-s)A}b(s)ds$$

since $(e^{tA})^{-1} = e^{-tA}$ and $e^{tA}e^{-sA} = e^{(t-s)A}$.

Another viewpoint

Suppose $A \in \mathbb{C}^{n\times n}$ is a constant diagonalizable matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and linearly independent eigenvectors $v_1, \ldots, v_n$. Then $\varphi_j(t) \equiv e^{\lambda_j t}v_j$ is a solution of (LHC) $x' = Ax$ since

$$\begin{aligned}\varphi_j' &= \frac{d}{dt}(e^{\lambda_j t}v_j) = \lambda_j e^{\lambda_j t}v_j = e^{\lambda_j t}(\lambda_j v_j) \\ &= e^{\lambda_j t}Av_j = A(e^{\lambda_j t}v_j) = A\varphi_j.\end{aligned}$$

Clearly $\varphi_1, \ldots, \varphi_n$ are linearly independent at $t = 0$ as $\varphi_j(0) = v_j$. Thus

$$\Phi(t) = [\varphi_1(t)\, \varphi_2(t) \cdots \varphi_n(t)]$$

is a F.M. for (LHC). So the general solution of (LHC) (for diagonalizable A) is $\Phi(t)c = c_1e^{\lambda_1 t}v_1 + \cdots + c_ne^{\lambda_n t}v_n$ for arbitrary scalars $c_1, \ldots, c_n$.

Remark on Exponentials

Let $B(t)$ be a C^1 $n \times n$ matrix function of t, and let $A(t) = B'(t)$. Then

$$\begin{aligned}\frac{d}{dt}(e^{B(t)}) &= \frac{d}{dt}(I + B + \frac{1}{2!}B\cdot B + \frac{1}{3!}B\cdot B\cdot B + \cdots) \\ &= A + \frac{1}{2!}(AB + BA) + \frac{1}{3!}(AB^2 + BAB + B^2A) + \cdots .\end{aligned}$$

Now, if for each t, $A(t)$ and $B(t)$ commute, then

$$\frac{d}{dt}(e^{B(t)}) = A\left(I + B + \frac{1}{2!}B^2 + \cdots\right) = B'(t)e^{B(t)}.$$

Now suppose we start with a continuous $n \times n$ matrix function $A(t)$, and for some t_0, we define $B(t) = \int_{t_0}^{t} A(s)ds$, so $B'(t) = A(t)$. Suppose in addition that $A(t)$ and $B(t)$ commute for all t. Then $\Phi(t) \equiv \exp\left(\int_{t_0}^{t} A(s)ds\right)$ is the F.M. for (LH) $x' = A(t)x$, normalized at t_0, since $\Phi(t_0) = I$ and $\Phi'(t) = A(t)\Phi(t)$ as above.

Remark. A sufficient (but not necessary) condition guaranteeing that $A(t)$ and $\int_{t_0}^{t} A(s)ds$ commute is that $A(t)$ and $A(s)$ commute for all t, s.

Application to Nonlinear Solution Operator

Consider the nonlinear DE $x' = f(t,x)$ where f is C^1, and let S_τ^t denote the solution operator. For a fixed τ, let $x(t,y)$ denote the solution of the IVP $x' = f(t,x)$, $x(\tau) = y$. The equation of variation for the $n \times n$ Jacobian matrix $D_y x$ is

$$\frac{d}{dt}\left(D_y x(t,y)\right) = \left(D_x f\left(t, x(t,y)\right)\right)\left(D_y x(t,y)\right),$$

and thus

$$\frac{d}{dt}\left(\det\ (D_y x(t,y))\right) = \operatorname{tr}\ \left(D_x f\left(t, x(t,y)\right)\right) \det\ \left(D_y x(t,y)\right).$$

This relation will be used and interpreted below. Solving, one obtains

$$\begin{aligned}\det\ (D_y x(t,y)) &= \det\ (D_y x(\tau,y)) \exp\left(\int_\tau^t \operatorname{tr}\ \left(D_x f\left(s, x(s,y)\right)\right) ds\right)\\ &= \exp\left(\int_\tau^t \operatorname{tr}\ \left(D_x f\left(s, x(s,y)\right)\right) ds\right),\end{aligned}$$

since

$$D_y x(\tau, y) = D_y y = I.$$

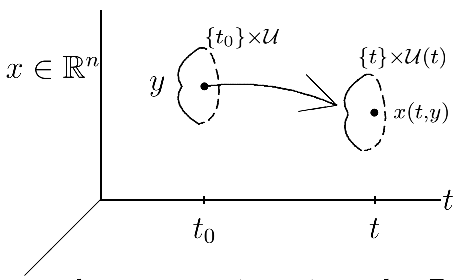
In particular, $\det\ (D_y x(t,y)) \neq 0$, so $D_y x(t,y)$ is invertible. For τ and t fixed, $D_y x(t,y) = D_y S_\tau^t$, so we have demonstrated again that $D_y S_\tau^t$ is invertible at each y.

Rate of Change of Volume in a Flow

Consider an autonomous system $x' = f(x)$, where f is C^1 and $\mathbb{F} = \mathbb{R}$, so $x \in \mathbb{R}^n$. Fix t_0, and view the family of IVPs

$$x' = f(x), \quad x(t_0) = y$$

for y in an open set $\mathcal{U} \subset \mathbb{R}^n$ as a flow: at the initial time t_0, there is a particle at each point $y \in \mathcal{U}$; that particle's location at time $t \geq t_0$ is given by $x(t,y)$, where $x(t,y)$ is the solution of the IVP $x' = f(x)$, $x(t_0) = y$ (e.g., f can be thought of as a steady-state velocity field). For $t \geq t_0$, let $\mathcal{U}(t) = \{x(t,y) : y \in \mathcal{U}\}$. Then $\mathcal{U}(t) = S_{t_0}^t(\mathcal{U})$ and $S_{t_0}^t : \mathcal{U} \to \mathcal{U}(t)$ is (for fixed t) a C^1 diffeomorphism (i.e., for fixed t, the map $y \mapsto x(t,y)$ is a C^1 diffeomorphism on $\mathcal{U}$). In particular, $\det D_y x(t,y)$ never vanishes. Assuming, in addition, that $\mathcal{U}$ is connected, $\det D_y x(t,y)$ must either be always positive or always negative; since $\det D_y x(t_0,y) = \det I = 1 > 0$, $\det D_y x(t,y)$ is always > 0.



Now the volume $\operatorname{vol}(\mathcal{U}(t))$ satisfies

$$\operatorname{vol}(\mathcal{U}(t)) = \int_{\mathcal{U}(t)} 1\, dx = \int_{\mathcal{U}} |\det D_y x(t,y)| dy = \int_{\mathcal{U}} \det D_y x(t,y) dy.$$

Assuming differentiation under the integral sign is justified (e.g., if $\mathcal{U}$ is contained in a compact set K and $S_{t_0}^t$ can be extended to $y \in K$), and using the relation derived in the previous section,

$$\begin{aligned}\frac{d}{dt}\left(\mathrm{vol}\left(\mathcal{U}(t)\right)\right) &= \int_{\mathcal{U}} \frac{d}{dt}\left(\det D_y x(t,y)\right) dy = \int_{\mathcal{U}} \mathrm{div} f(x(t,y)) \det D_y x(t,y) dy \\ &= \int_{\mathcal{U}(t)} \mathrm{div} f(x) dx,\end{aligned}$$

where the divergence of f is by definition

$$\mathrm{div} f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \cdots + \frac{\partial f_n}{\partial x_n} = \mathrm{tr}\,\left(D_x f(x)\right).$$

Thus the rate of change of the volume of $\mathcal{U}(t)$ is the integral of the divergence of f over $\mathcal{U}(t)$. In particular, if $\mathrm{div} f(x) \equiv 0$, then $\frac{d}{dt}\left(\mathrm{vol}\left(\mathcal{U}(t)\right)\right) = 0$, and volume is conserved.

Remark. The same argument applies when $f = f(t,x)$ depends on t as well: just replace $\mathrm{div} f(x)$ by $\mathrm{div}_x f(t,x)$, the divergence of f (with respect to x):

$$\mathrm{div}_x f(t,x) = \left.\left(\frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}\right)\right|_{(t,x)}.$$

Linear Systems with Periodic Coefficients

Let $A : \mathbb{R} \to \mathbb{C}^{n\times n}$ be continuous and periodic with period $\omega > 0$:

$$(\forall\, t \in \mathbb{R}) \qquad A(t+\omega) = A(t).$$

Note that in this case we take the scalar field to be $\mathbb{F} = \mathbb{C}$. Consider the periodic linear homogeneous system

$$\text{(PLH)} \qquad x' = A(t)x, \quad t \in \mathbb{R}.$$

All solutions exist for all time $t \in \mathbb{R}$ because the system is linear and A is defined and continuous for $t \in \mathbb{R}$.

Lemma. If $\Phi(t)$ is a F.M. for (PLH), then so also is $\Psi(t) \equiv \Phi(t+\omega)$.

Proof. For each t, $\Psi(t)$ is invertible. Also, $\Psi'(t) = \Phi'(t+\omega) = A(t+\omega)\Phi(t+\omega) = A(t)\Psi(t)$, so $\Psi(t)$ is a matrix solution of (PLH). □

Theorem. To each F.M. $\Phi(t)$ for (PLH), there exists an invertible periodic C^1 matrix function $P : \mathbb{R} \to \mathbb{C}^{n\times n}$ and a *constant* matrix $R \in \mathbb{C}^{n\times n}$ for which $\Phi(t) = P(t)e^{tR}$.

Proof. By the lemma, there is an invertible matrix $C \in \mathbb{C}^{n\times n}$ such that $\Phi(t+\omega) = \Phi(t)C$. Since C is invertible, it has a logarithm, i.e. there exists a matrix $W \in \mathbb{C}^{n\times n}$ such that $e^W = C$. Let $R = \frac{1}{\omega}W$. Then $C = e^{\omega R}$. Define $P(t) = \Phi(t)e^{-tR}$. Then $P(t)$ is invertible for all t, $P(t)$ is C^1, and $\Phi(t) = P(t)e^{tR}$. Finally,

$$\begin{aligned}P(t+\omega) &= \Phi(t+\omega)e^{-(t+\omega)R} \\ &= \Phi(t)Ce^{-\omega R}e^{-tR} = \Phi(t)e^{-tR} = P(t),\end{aligned}$$

so $P(t)$ is periodic. □

Linear Scalar n^{th}-order ODEs

Let $I \equiv [a,b]$ be an interval in $\mathbb{R}$, and suppose $a_j(t)$ are in $C(I,\mathbb{F})$ for $j = 0,1,\ldots,n$, with $a_n(t) \neq 0 \,\forall\, t \in I$. Consider the n^{th}-order linear differential operator $L : C^n(I) \to C(I)$ given by

$$Lu = a_n(t)\frac{d^n u}{dt^n} + \cdots + a_1(t)\frac{du}{dt} + a_0(t)u,$$

and the n^{th}-order homogeneous equation (nLH) $Lu = 0$, $t \in I$. Consider the equivalent $n \times n$ first-order system (LH) $x' = A(t)x$, $t \in I$, where

$$A(t) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \frac{-a_0}{a_n} & \cdots & & -\frac{a_{n-1}}{a_n} \end{bmatrix} \qquad \text{and} \qquad x = \begin{bmatrix} u \\ u' \\ u'' \\ \vdots \\ u^{(n-1)} \end{bmatrix} \in \mathbb{F}^n.$$

Fix $t_0 \in I$. Appropriate initial conditions for (nLH) are

$$\begin{bmatrix} u(t_0) \\ u'(t_0) \\ \vdots \\ u^{(n-1)}(t_0) \end{bmatrix} = x(t_0) = \zeta \equiv \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{bmatrix}.$$

Recall that u is a C^n solution of (nLH) if and only if x is a C^1 solution of (LH), with a similar equivalence between associated IVP's. If $\Phi(t)$ is a F.M. for (LH), with $A(t)$ as given above, then $\Phi(t)$ has the form

$$\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{bmatrix},$$

where each $\varphi_j(t)$ satisfies (nLH).

Definition. If $\varphi_1(t),\ldots,\varphi_n(t)$ are solutions of (nLH), then the *Wronskian* of $\varphi_1,\ldots,\varphi_n$ (a scalar function of t) is defined to be

$$W(\varphi_1,\ldots,\varphi_n)(t) = \det \begin{bmatrix} \varphi_1(t) & & \varphi_n(t) \\ \varphi_1'(t) & \cdots & \varphi_n'(t) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(t) & & \varphi_n^{(n-1)}(t) \end{bmatrix} \;(= \det\Phi(t)).$$

Since $\Phi(t)$ is a matrix solution of (LH), we know

$$\det(\Phi(t)) = \det(\Phi(t_0)) \exp\int_{t_0}^{t} \operatorname{tr}(A(s))ds,$$

so

$$W(\varphi_1, \ldots, \varphi_n)(t) = W(\varphi_1, \ldots, \varphi_n)(t_0) \exp \int_{t_0}^{t} -\frac{a_{n-1}(s)}{a_n(s)} ds.$$

In particular, for solutions $\varphi_1, \ldots, \varphi_n$ of (nLH),

$$\text{either} \quad W(\varphi_1, \ldots, \varphi_n)(t) \equiv 0 \text{ on } I, \text{ or } (\forall\, t \in I) \quad W(\varphi_1, \ldots, \varphi_n)(t) \neq 0.$$

Theorem. Let $\varphi_1, \ldots, \varphi_n$ be n solutions of (nLH) $Lu = 0$. Then they are linearly independent on I (i.e., as elements of $C^n(I)$) if and only if $W(\varphi_1, \ldots, \varphi_n)(t) \neq 0$ on I.

Proof. If $\varphi_1, \ldots, \varphi_n$ are linearly dependent in $C^n(I)$, then there exist scalars $c_1, \ldots, c_n$ such that

$$c_1\varphi_1(t) + \cdots + c_n\varphi_n(t) \equiv 0 \text{ on } I, \text{ with } c \equiv \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \neq 0;$$

thus $\Phi(t)c = 0$ on I, so $W(\varphi_1, \ldots, \varphi_n)(t) = \det \Phi(t) = 0$ on I. Conversely, if $\det \Phi(t) = 0$ on I, then the solutions

$$\begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_1^{(n-1)} \end{bmatrix}, \ldots, \begin{bmatrix} \varphi_n \\ \vdots \\ \varphi_n^{(n-1)} \end{bmatrix}$$

of (LH) are linearly dependent (as elements of $C^1(I, \mathbb{F}^n)$), so there exist scalars $c_1, \ldots, c_n$ such that

$$c_1 \begin{bmatrix} \varphi_1(t) \\ \vdots \end{bmatrix} + \cdots + c_n \begin{bmatrix} \varphi_n(t) \\ \vdots \end{bmatrix} \equiv 0 \text{ on } I,$$

where not all $c_j = 0$. In particular, $c_1\varphi_1(t) + \cdots + c_n\varphi_n(t) \equiv 0$ on I, so $\varphi_1, \ldots, \varphi_n$ are linearly dependent in $C^n(I)$. □

Corollary. The dimension of the vector space of solutions of (nLH) (a subspace of $C^n(I)$) is n, i.e., $\dim \mathcal{N}(L) = n$, where $\mathcal{N}(L)$ denotes the null space of $L : C^n(I) \to C(I)$.

The differential operator L (normalized so that $a_n(t) \equiv 1$) is itself determined by n linearly independent solutions of (nLH) $Lu = 0$:

Fact. Suppose $\varphi_1(t), \ldots, \varphi_n(t) \in C^n(I)$ with $W(\varphi_1, \ldots, \varphi_n)(t) \neq 0$ $(\forall\, t \in I)$. Then there exists a unique n^{th} order linear differential operator

$$L = \frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t)$$

(with $a_n(t) \equiv 1$ and each $a_j(t) \in C(I)$) for which $\varphi_1, \ldots, \varphi_n$ form a fundamental set of solutions of (nLH) $Lu = 0$, namely,

$$Lu = \frac{W(\varphi_1, \ldots, \varphi_n, u)}{W(\varphi_1, \ldots, \varphi_n)}$$

where

$$W(\varphi_1, \ldots, \varphi_n, u) = \det \begin{bmatrix} \varphi_1 & \cdots & \varphi_n & u \\ \varphi_1' & & \varphi_n' & u' \\ \vdots & & \vdots & \vdots \\ \varphi_1^{(n)} & \cdots & \varphi_n^{(n)} & u^{(n)} \end{bmatrix}.$$

Sketch. In this formula for Lu, expanding the determinant in the last column shows that L is an n^{th} order linear differential operator with continuous coefficients $a_j(t)$ and $a_n(t) \equiv 1$. Clearly $\varphi_1, \ldots, \varphi_n$ are solutions of $Lu = 0$. For uniqueness (with $a_n(t) \equiv 1$), note that if $\varphi_1, \ldots, \varphi_n$ are linearly independent solutions of $Lu = 0$ for some L, then

$$\Phi^T(t) \begin{bmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_{n-1}(t) \end{bmatrix} = - \begin{bmatrix} \varphi_1^{(n)}(t) \\ \vdots \\ \varphi_n^{(n)}(t) \end{bmatrix}.$$

Since $W(\varphi_1, \ldots, \varphi_n)(t) \neq 0$ $(\forall\, t \in I)$, $\Phi(t)$ is invertible $\forall\, t \in I$, so

$$\begin{bmatrix} a_0(t) \\ \vdots \\ a_{n-1}(t) \end{bmatrix} = -(\Phi^T)^{-1}(t) \begin{bmatrix} \varphi_1^{(n)}(t) \\ \vdots \\ \varphi_n^{(n)}(t) \end{bmatrix}$$

is uniquely determined by $\varphi_1, \ldots, \varphi_n$.

Remark. A first-order system (LH) $x' = A(t)x$ is uniquely determined by any F.M. $\Phi(t)$. Since $\Phi'(t) = A(t)\Phi(t)$, $A(t) = \Phi'(t)\Phi^{-1}(t)$.

Linear Inhomogeneous n^{th}-order scalar equations

For simplicity, normalize the coefficients $a_j(t)$ so that $a_n(t) \equiv 1$ in L. Consider

$$\text{(nLI)} \qquad Lu = u^{(n)} + a_{n-1}(t)u^{(n-1)} + \cdots + a_0(t)u = \beta(t).$$

Let

$$x = \begin{bmatrix} u \\ u' \\ \vdots \\ u^{(n-1)} \end{bmatrix}, \quad b(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \beta(t) \end{bmatrix}, \quad \text{and} \quad A(t) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & & \cdots & -a_{n-1} \end{bmatrix};$$

then $x(t)$ satisfies (LI) $x' = A(t)x + b(t)$. We can apply our results for (LI) to obtain expressions for solutions of (nLI).

Theorem. If $\varphi_1, \ldots, \varphi_n$ is a fundamental set of solutions of (nLH) $Lu = 0$, then the solution $\psi(t)$ of (nLI) $Lu = \beta(t)$ with initial condition $u^{(k)}(t_0) = \zeta_{k+1}$ $(k = 0, \ldots, n-1)$ is

$$\psi(t) = \varphi(t) + \sum_{k=1}^{n} \varphi_k(t) \int_{t_0}^{t} \frac{W_k(\varphi_1, \ldots, \varphi_n)(s)}{W(\varphi_1, \ldots, \varphi_n)(s)} \beta(s) ds$$

where $\varphi(t)$ is the solution of (nLH) with the same initial condition at t_0, and W_k is the determinant of the matrix obtained from

$$\Phi(t) = \begin{bmatrix} \varphi_1 & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_n' \\ \vdots & & \vdots \\ \varphi_1^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{bmatrix}$$

by replacing the k^{th} column of $\Phi(t)$ by the n-th unit coordinate vector e_n.

Proof. We know

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)b(s)ds,$$

where $x_0 = [\zeta_1, \ \cdots \ \zeta_n]^T$ and $b(s) = [0 \ \cdots \ \beta(s)]^T$, solves the IVP $x' = A(t)x, \ x(t_0) = x_0$. The first component of $x(t)$ is $\psi(t)$, and the first component of $\Phi(t)\Phi^{-1}(t_0)x_0$ is the solution $\varphi(t)$ of (nLH) described above. By Cramer's Rule, the k^{th} component of $\Phi^{-1}(s)e_n$ is

$$\frac{W_k(\varphi_1, \ldots, \varphi_n)(s)}{W(\varphi_1, \ldots, \varphi_n)(s)}.$$

Thus the first component of $\Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)b(s)ds$ is

$$[\varphi_1(t) \cdots \varphi_n(t)] \int_{t_0}^{t} \Phi^{-1}(s)e_n\beta(s)ds = \sum_{k=1}^{n} \varphi_k(t) \int_{t_0}^{t} \frac{W_k(\varphi_1, \ldots, \varphi_n)(s)}{W(\varphi_1, \ldots, \varphi_n)(s)}\beta(s)ds.$$

□

Linear n^{th}-order scalar equations with constant coefficients

For simplicity, take $a_n = 1$ and $\mathbb{F} = \mathbb{C}$. Consider

$$Lu = u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_0 u,$$

where $a_0, \ldots, a_{n-1}$ are constants. Then

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & & \cdots & -a_{n-1} \end{bmatrix}$$

has characteristic polynomial

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

Moreover, since A is a companion matrix, each distinct eigenvalue of A has only one Jordan block in the Jordan form of A. Indeed, for any λ,

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ -a_0 & & & (-a_{n-1} - \lambda) \end{bmatrix}$$

has rank $\geq n-1$, so the geometric multiplicity of each eigenvalue is $1 = \dim(\mathcal{N}(A - \lambda I))$.

Now if λ_k is a root of $p(\lambda)$ having multiplicity m_k (as a root of $p(\lambda)$), then terms of the form $t^j e^{\lambda_k t}$ for $0 \leq j \leq m_k - 1$ appear in elements of e^{tJ} (where $P^{-1}AP = J$ is in Jordan form), and thus also appear in $e^{tA} = Pe^{tJ}P^{-1}$, the F.M. for (LH) $x' = Ax$, normalized at 0. This explains the well-known result:

Theorem. Let $\lambda_1, \ldots, \lambda_s$ be the distinct roots of $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$, and suppose λ_k has multiplicity m_k for $1 \leq k \leq s$. Then a fundamental set of solutions of

$$Lu = u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_0 u = 0,$$

where $a_k \in \mathbb{C}$, is

$$\{t^j e^{\lambda_k t} : 1 \leq k \leq s, 0 \leq j \leq m_k - 1\}.$$

The standard proof is to show that these functions are linearly independent and then plug in and verify that they are solutions: write

$$L = \left(\frac{d}{dt} - \lambda_1\right)^{m_1} \cdots \left(\frac{d}{dt} - \lambda_s\right)^{m_s},$$

and use

$$\left(\frac{d}{dt} - \lambda_k\right)^{m_k} \left(t^j e^{\lambda_k t}\right) = 0 \quad \text{for} \quad 0 \leq j \leq m_k - 1.$$