Fourier Series

Let $\{e_j : 1 \le j \le n\}$ be the standard basis in $\mathbb{R}^n$. We say $f : \mathbb{R}^n \to \mathbb{C}$ is 2π-periodic in each variable if

$$f(x + 2\pi e_j) = f(x) \qquad \forall\, x \in \mathbb{R}^n,\ 1 \le j \le n.$$

We can identify 2π-periodic functions with functions on a torus. Let $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\} \subset \mathbb{C}$, and $T^n = S^1 \times \cdots \times S^1 \subset \mathbb{C}^n$. To each function $\widetilde{\phi} : T^n \to \mathbb{C}$ we can identify a 2π-periodic function $\phi : \mathbb{R}^n \to \mathbb{C}$ by $\phi(x_1, \ldots, x_n) = \widetilde{\phi}(e^{ix_1}, \ldots, e^{ix_n})$. Conversely, each 2π-periodic function $\phi : \mathbb{R}^n \to \mathbb{C}$ induces a unique $\widetilde{\phi} : T^n \to \mathbb{C}$ for which $\widetilde{\phi}(e^{ix_1}, \ldots, e^{ix_n}) = \phi(x_1, \ldots, x_n)$. If $\phi : \mathbb{R}^n \to \mathbb{C}$ is 2π-periodic, ϕ is uniquely determined by its values $\phi(x)$ for $x \in [-\pi, \pi)^n$ or for $x \in [0, 2\pi)^n$. Let $\nu_n = (2\pi)^{-n}\lambda_n$, where λ_n is n-dimensional Lebesgue measure. Then ν_n induces a measure $\widetilde{\nu}_n$ on T^n for which

$$\int_{T^n} \widetilde{\phi}\, d\widetilde{\nu}_n = \int_{[0,2\pi]^n} \phi\, d\nu_n.$$

From here on, we blur the distinction between ϕ and $\widetilde{\phi}$ and between ν_n and $\widetilde{\nu}_n$, and we will abuse these notations. Note: $\nu_n(T^n) = \nu_n([0, 2\pi]^n) = 1$. Let $L^p(T^n)$ denote $L^p([0, 2\pi]^n)$ with measure ν_n $(1 \le p \le \infty)$. $L^2(T^n)$ is a Hilbert space with inner product

$$(\psi, \phi) = \int_{T^n} \bar{\psi}\phi\, d\nu_n = \int_{[0,2\pi]^n} \bar{\psi}\phi\, d\nu_n.$$

Theorem. $\{e^{ix\cdot\xi} : \xi \in \mathbb{Z}^n\}$ is an orthonormal system in $L^2(T^n)$.

Proof.

$$(e^{ix\cdot\eta}, e^{ix\cdot\xi}) = \int_{[0,2\pi]^n} e^{ix\cdot(\xi-\eta)}\, d\nu_n = \begin{cases} 1, & \xi = \eta \\ 0, & \xi \ne \eta \end{cases}.$$

$\square$

Definition. A *trigonometric polynomial* is a finite linear combination of $\{e^{ix\cdot\xi} : \xi \in \mathbb{Z}^n\}$.

Note: since $\{e^{ix\cdot\xi}, e^{-ix\cdot\xi}\}$ and $\{\cos(x\cdot\xi), \sin(x\cdot\xi)\}$ span the same two-dimensional space, we could use sines and cosines as our basic functions.

Definition. $C(T^n)$ is the space of all continuous 2π-periodic functions $\phi : \mathbb{R}^n \to \mathbb{C}$. Note that $C(T^n) \subsetneqq C([0, 2\pi]^n)$.

We will use the uniform norm $\|\phi\|_u = \sup_x |\phi(x)|$ on $C(T^n)$. $C^k(T^n)$ (for $k \ge 0, k \in \mathbb{Z}$) is the space of all C^k 2π-periodic functions $\phi : \mathbb{R}^n \to \mathbb{C}$. Again $C^k(T^n) \subsetneqq C^k([0, 2\pi]^n)$. We will use the norm $\|\phi\|_{C^k} = \sum_{|\alpha| \le k} \|\partial^\alpha \phi\|_u$.

Fourier Coefficients

For $f \in L^1(T^n)$, define

$$\widehat{f}(\xi) = \int_{T^n} e^{-ix\cdot\xi} f(x) d\nu_n(x)$$

for $\xi \in \mathbb{Z}^n$. Then $|\widehat{f}(\xi)| \leq \|f\|_1$. We regard $\widehat{f}$ as a map $\widehat{f} : \mathbb{Z}^n \to \mathbb{C}$; then $\widehat{f} \in l^\infty(\mathbb{Z}^n)$ and $\|\widehat{f}\|_\infty \leq \|f\|_1$. Since $\nu(T^n) = 1$, we have $L^2(T^n) \subset L^1(T^n)$ and $\|f\|_1 \leq \|f\|_2$. For $f \in L^2(T^n)$, $\widehat{f}$ can be expressed as an inner product: $\widehat{f}(\xi) = (e^{ix\cdot\xi}, f)$.

The *Fourier series* of f is the *formal series* $\sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi) e^{ix\cdot\xi}$. We will study in what sense this series converges to f. A key role is played by what is called a *summability kernel*; this is a sequence Q_k with properties (1), (2), (3), (4) below. Define

$$Q_k(x) = a_k^{-1} \left(\prod_{j=1}^n \tfrac{1}{2}(1 + \cos x_j) \right)^k$$

where

$$a_k = \int_{T^n} \left(\prod_{j=1}^n \tfrac{1}{2}(1 + \cos x_j) \right)^k d\nu_n(x).$$

Lemma.

(1) Q_k is a trigonometric polynomial.

(2) $Q_k(x) \geq 0$

(3) $\int_{T^n} Q_k(x) d\nu_n(x) = 1$.

(4) For $0 < \delta < \pi$, set

$$\eta_k(\delta) = \max\{Q_k(x) : x \in [-\pi, \pi]^n \backslash (-\delta, \delta)^n\}.$$

Then $\lim_{k\to\infty} \eta_k(\delta) = 0$.

Proof. The first three properties are clear. To prove (4), we first show that the sequence a_k satisfies $a_k \geq (2(k+1))^{-n}$. In fact, since $1 + \cos x$ is non-negative on $[0, \pi]$ and concave on $[0, \frac{\pi}{2}]$, we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\tfrac{1}{2}(1 + \cos x)\right)^k dx &= \frac{1}{\pi} \int_0^{\pi} \left(\tfrac{1}{2}(1 + \cos x)\right)^k dx \\ &\geq \frac{1}{\pi} \int_0^{\pi/2} \left(1 - \frac{x}{\pi}\right)^k dx \\ &= (k+1)^{-1}(1 - 2^{-k-1}) \geq (2(k+1))^{-1}. \end{aligned}$$

Now if $x \in [-\pi, \pi]^n \backslash (-\delta, \delta)^n$, then

$$Q_k(x) \leq a_k^{-1} \left(\tfrac{1}{2}(1 + \cos\delta)\right)^k \leq (2(k+1))^n \left(\tfrac{1}{2}(1 + \cos\delta)\right)^k \to 0$$

as $k \to \infty$. $\square$

Theorem. Given $f \in C(T^n)$, let

$$p_k(x) = (f * Q_k)(x) = \int_{T^n} f(x-y)Q_k(y)d\nu_n(y).$$

Then p_k is a trigonometric polynomial, and $\|p_k - f\|_u \to 0$ as $k \to \infty$.

Proof. Since $\widehat{Q}_k(\xi) = 0$ for $|\xi|$ sufficiently large,

$$\begin{aligned} p_k(x) &= \int_{T^n} f(y)Q_k(x-y)d\nu_n(y) = \int_{T^n} f(y)\sum_\xi \widehat{Q}_k(\xi)e^{i(x-y)\cdot\xi}d\nu_n(y) \\ &= \sum_\xi \widehat{f}(\xi)\widehat{Q}_k(\xi)e^{ix\cdot\xi} \end{aligned}$$

is a trigonometric polynomial. Given $\epsilon > 0$, the uniform continuity of f implies that there is $\delta > 0$ such that

$$|x_1 - x_2|_\infty < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon.$$

Then

$$\begin{aligned} |p_k(x) - f(x)| &= \left|\int_{T^n} (f(x-y) - f(x))Q_k(y)d\nu_n(y)\right| \\ &\le \int_{T^n} |f(x-y) - f(x)|Q_k(y)d\nu_n(y) = I_1 + I_2, \end{aligned}$$

where I_1 is the integral over $y \in (-\delta,\delta)^n$ and I_2 is the integral over $y \in [-\pi,\pi]^n \backslash (-\delta,\delta)^n$. Now

$$\begin{aligned} I_1 &\le \int_{(-\delta,\delta)^n} \epsilon\, Q_k(y)\, d\nu_n(y) \le \epsilon, \text{ and} \\ I_2 &\le \int_{[-\pi,\pi]^n\backslash(-\delta,\delta)^n} 2\|f\|_u\, \eta_k(\delta)\, d\nu_n \le 2\|f\|_u\, \eta_k(\delta) < \epsilon \end{aligned}$$

for k sufficiently large, so the result follows. $\square$

Corollary. Trigonometric polynomials are dense in $C(T^n)$.

Remark. The proof of the Theorem above used only the properties (1)–(4) of the Q_k. Therefore the same result holds for any summability kernel. Another example for $n = 1$ is the Féjer kernel, which is given by

$$F_k(x) = (k+1)^{-1}\frac{\sin^2\left(\frac{1}{2}(k+1)x\right)}{\sin^2\left(\frac{1}{2}x\right)} = \sum_{\xi=-k}^{k}\left(1 - \frac{|\xi|}{k+1}\right)e^{ix\xi}.$$

If we define $S_k(f) = \sum_{\xi=-k}^{k} \widehat{f}(\xi)e^{ix\cdot\xi}$ and $\sigma_k(f) = (k+1)^{-1}(S_0(f) + \cdots + S_k(f))$, then $\sigma_k(f) = f * F_k$. It follows that for $f \in C(T)$, $\sigma_k(f) \to f$ uniformly. This is the classical result that the Fourier series of $f \in C(T)$ is Cesàro summable to f.

The partial sums $S_k(f)$ themselves are obtained by convolving f with the "Dirichlet kernel": $S_k(f) = f * D_k$, where

$$D_k(x) = \sum_{\xi=-k}^{k} e^{ix\xi} = \frac{\sin\left(\left(k+\frac{1}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)}.$$

The Dirichlet kernel however, is *not* a summability kernel: D_k is not nonnegative (not horrible), and more crucially it does not satisfy condition (4) of a summability kernel. This is the reason that pointwise convergence of Fourier series is a delicate matter.

Corollary. Trigonometric polynomials are dense in $L^2(T^n)$.

Proof. Given $f \in L^2(T^n)$ and $\epsilon > 0$, there exists $g \in C(T^n)$ such that $\|f-g\|_2 < \frac{1}{2}\epsilon$ and there exists a trigonometric polynomial p such that $\|p-g\|_u < \frac{1}{2}\epsilon$, so since $\nu_n(T^n) = 1$,

$$\|f-p\|_2 \leq \|f-g\|_2 + \|g-p\|_2 \leq \|f-g\|_2 + \|g-p\|_u < \epsilon.$$

□

Corollary. $\{e^{ix\cdot\xi} : \xi \in \mathbb{Z}^n\}$ is a complete orthonormal system in $L^2(T^n)$.

Hence if $f \in L^2(T^n)$, the Fourier series of f (any arrangement) converges to f in L^2. Also, the map $\mathcal{F} : L^2(T^n) \to l^2(\mathbb{Z}^n)$ given by $f \mapsto \widehat{f}$ is a Hilbert space isomorphism and $\|\widehat{f}\|_{l^2(\mathbb{Z}^n)} = \|f\|_{L^2(T^n)}$.

Theorem. If $f \in L^1(T^n)$, then $p_k \to f$ in $L^1(T^n)$, where $p_k = f * Q_k$ and Q_k is as above.

Proof. The proof is similar to the proof of the Theorem above, except we use continuity of translation in L^1 instead of uniform continuity. Given $\epsilon > 0$, choose $\delta \ni \|f(x-\alpha)-f(x)\|_1 < \epsilon$ whenever $|\alpha|_\infty < \delta$. By Fubini,

$$\int_{T^n} |p_k(x) - f(x)| d\nu_n(x) \leq \int_{T^n} \left[Q_k(y) \int_{T^n} |f(x-y) - f(x)| d\nu_n(x) \right] d\nu_n(y) = I_1 + I_2,$$

and

$$\begin{aligned} I_1 &\leq \int Q_k(y)\, \epsilon \, d\nu_n(y) = \epsilon \\ I_2 &\leq 2\|f\|_1 \, \eta_k(\delta) \to 0 \text{ as } k \to \infty. \end{aligned}$$

□

Corollary. (Uniqueness Theorem). If $f \in L^1(T^n)$ and $(\forall\, \xi \in \mathbb{Z}^n)\widehat{f}(\xi) = 0$, then $f = 0$ a.e. Thus if $f, g \in L^1(T^n)$ and $\widehat{f} \equiv \widehat{g}$, then $f = g$ a.e.

Proof. If $\widehat{f} \equiv 0$, then $p_k(x) = \sum_\xi \widehat{f}(\xi)\widehat{Q}_k(\xi)e^{ix\cdot\xi} = 0$, and $p_k \to f$ in L^1. □

Theorem. (Riemann-Lebesgue Lemma). If $f \in L^1(T^n)$, then $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Proof. This follows from the analogous result for the Fourier transform, which will be proved later. The statement for the Fourier transform is that if $f \in L^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx \to 0$ as $|\xi| \to \infty$, $\xi \in \mathbb{R}^n$. The Fourier series version stated here follows by applying the $\mathbb{R}^n$ version to $f(x)\chi_{[-\pi,\pi]^n}(x) \in L^1(\mathbb{R}^n)$ and restricting to $\xi \in \mathbb{Z}^n$. □

Absolutely Convergent Fourier Series

Theorem. Suppose $f \in L^1(T^n)$ and $\widehat{f} \in l^1(\mathbb{Z}^n)$. Then the Fourier series of f converges absolutely and uniformly to a $g \in C(T^n)$, and $g = f$ a.e.

Proof. Let $g(x) = \sum_{\xi\in\mathbb{Z}^n} \widehat{f}(\xi)e^{ix\cdot\xi}$. Since $\widehat{f} \in l^1(\mathbb{Z}^n)$, this series converges uniformly and absolutely, and $g \in C(T^n)$. By the Dominated Convergence Theorem,

$$\begin{aligned}
\widehat{g}(\xi) &= \int_{T^n} e^{-ix\cdot\xi}\left(\sum_{\eta\in\mathbb{Z}^n} \widehat{f}(\eta)e^{ix\cdot\eta}\right) d\nu_n(x) \\
&= \sum_{\eta\in\mathbb{Z}^n} \widehat{f}(\eta)\int_{T^n} e^{-ix\cdot\xi}e^{ix\cdot\eta}d\nu_n(x) = \widehat{f}(\xi),
\end{aligned}$$

so $g = f$ a.e. □

Decay of Fourier Coefficients $\leftrightarrow$ Smoothness of f

Lemma. Suppose $\alpha(\xi) \in l^1(\mathbb{Z}^n)$ and $i\xi_j\alpha(\xi) \in l^1(\mathbb{Z}^n)$. Let $f = \sum_\xi \alpha(\xi)e^{ix\cdot\xi}$ and $g = \sum_\xi i\xi_j\alpha(\xi)e^{ix\cdot\xi}$. Then $f, g \in C(T^n)$, $\partial_{x_j} f$ exists everywhere, and $\partial_{x_j} f = g$.

Proof. The two series of continuous functions converge absolutely and uniformly to f and g, respectively. Since $\partial_{x_j}(\alpha(\xi)e^{ix\cdot\xi}) = i\xi_j\alpha(\xi)e^{ix\cdot\xi}$, the result follows from a standard theorem in analysis. □

Theorem. Suppose $f \in L^1(T^n)$ and $(1 + |\xi|^m)\widehat{f}(\xi) \in l^1(\mathbb{Z}^n)$ for some integer $m \geq 0$. Then the Fourier series of f converges absolutely and uniformly to a $g \in C^m(T^n)$, and $f = g$ a.e.

Proof. By the theorem above, we only have to show that $g \in C^m(T^n)$. For each ν with $|\nu| \leq m$, $(i\xi)^\nu\widehat{f}(\xi) \in l^1(\mathbb{Z}^n)$, so $\sum_\xi (i\xi)^\nu\widehat{f}(\xi)e^{ix\cdot\xi}$ converges absolutely and uniformly to some $g_\nu \in C(T^n)$. By the Lemma and induction, $g_\nu = \partial^\nu g$. □

Theorem. Suppose $f \in C^m(T^n)$.

(a) For $|\nu| \leq m$, $\widehat{\partial_x^\nu f}(\xi) = (i\xi)^\nu\widehat{f}(\xi)$.

(b) $(1 + |\xi|^m)\widehat{f}(\xi) \in l^2(\mathbb{Z}^n)$ and $\|(1 + |\xi|^m)\widehat{f}(\xi)\|_{l^2(\mathbb{Z}^n)} \leq c_{n,m}\|f\|_{C^m(T^n)}$ for some constant $c_{n,m}$ depending only on n, m.

(c) $|\widehat{f}(\xi)| \leq c_{n,m}(1 + |\xi|)^{-m}\|f\|_{C^m(T^n)}$ for $\xi \in \mathbb{Z}^n$.

Proof.

(a) follows by integration by parts. Set $x' = (x_2, \ldots, x_n)$, so $x = (x_1, x')$. Then

$$\begin{aligned} \widehat{\frac{\partial f}{\partial x_1}}(\xi) &= \int_{T^{n-1}} \left[\int_T e^{-ix\cdot\xi} \frac{\partial f}{\partial x_1}(x)\, d\nu_1(x_1) \right] d\nu_{n-1}(x') \\ &= \int_{T^{n-1}} \left[(i\xi_1) \int_T e^{-ix\cdot\xi} f(x)\, d\nu_1(x_1) \right] d\nu_{n-1}(x') \\ &= (i\xi_1) \widehat{f}(\xi). \end{aligned}$$

Iterate for higher derivatives.

(b) Part (a) gives for $1 \le j \le n$:

$$\|\xi_j^m \widehat{f}\|_{l^2(\mathbb{Z}^n)} = \|\widehat{\partial_{x_j}^m f}\|_{l^2(\mathbb{Z}^n)} = \|\partial_{x_j}^m f\|_{L^2(T^n)} \le \|\partial_{x_j}^m f\|_u.$$

Adding gives

$$\|(1 + |\xi_1|^m + \cdots + |\xi_n|^m)|\widehat{f}(\xi)|\|_{l^2(\mathbb{Z}^n)} \le \|f\|_{C^m(T^n)},$$

and (b) follows.

(c) is immediate from (b) and the fact that $\|\cdot\|_\infty \le \|\cdot\|_2$ on functions on $\mathbb{Z}^n$. $\square$

In comparing the last two theorems, we see that in order to conclude that a given $f \in L^1(T^n)$ is C^m, it suffices to know that $(1+|\xi|)^m \widehat{f} \in l^1$, and in the other direction, if f is C^m, then $(1+|\xi|)^m \widehat{f} \in l^2$. Thus the space of Fourier coefficients of C^m functions lies between $(1+|\xi|)^{-m} l^1$ and $(1+|\xi|)^{-m} l^2$. So faster decay of $\widehat{f}$ corresponds to more smoothness of f.