Ordinary Differential Equations

Existence and Uniqueness Theory

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Throughout this discussion, $|\cdot|$ will denote the Euclidean norm (i.e ℓ^2-norm) on $\mathbb{F}^n$ (so $\|\cdot\|$ is free to be used for norms on function spaces). An ordinary differential equation (ODE) is an equation of the form

$$g(t, x, x', \ldots, x^{(m)}) = 0$$

where g maps a subset of $\mathbb{R} \times (\mathbb{F}^n)^{m+1}$ into $\mathbb{F}^n$. A *solution* of this ODE on an interval $I \subset \mathbb{R}$ is a function $x : I \to \mathbb{F}^n$ for which $x', x'', \ldots, x^{(m)}$ exist at each $t \in I$, and

$$(\forall\, t \in I) \qquad g(t, x(t), x'(t), \ldots, x^{(m)}(t)) = 0\ .$$

We will focus on the case where $x^{(m)}$ can be solved for explicitly, i.e., the equation takes the form

$$x^{(m)} = f(t, x, x', \ldots, x^{(m-1)}),$$

and where the function f mapping a subset of $\mathbb{R} \times (\mathbb{F}^n)^m$ into $\mathbb{F}^n$ is continuous. This equation is called an m^{th}-*order* $n \times n$ system of ODE's. Note that if x is a solution defined on an interval $I \subset \mathbb{R}$ then the existence of $x^{(m)}$ on I (including one-sided limits at the endpoints of I) implies that $x \in C^{m-1}(I)$, and then the equation implies $x^{(m)} \in C(I)$, so $x \in C^m(I)$.

Reduction to First-Order Systems

Every m^{th}-order $n \times n$ system of ODE's is equivalent to a first-order $mn \times mn$ system of ODE's. Defining

$$y_j(t) = x^{(j-1)}(t) \quad \in \mathbb{F}^n \quad \text{for} \quad 1 \le j \le m$$

and

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \in \mathbb{F}^{mn},$$

the system

$$x^{(m)} = f(t, x, \ldots, x^{(m-1)})$$

is equivalent to the first-order $mn \times mn$ system

$$y' = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \\ f(t, y_1, \ldots, y_m) \end{bmatrix}$$

(see problem 1 on Problem Set 9).

Relabeling if necessary, we will focus on first-order $n \times n$ systems of the form $x' = f(t, x)$, where f maps a subset of $\mathbb{R} \times \mathbb{F}^n$ into $\mathbb{F}^n$ and f is continuous.

Example: Consider the $n \times n$ system $x'(t) = f(t)$ where $f : I \to \mathbb{F}^n$ is continuous on an interval $I \subset \mathbb{R}$. (Here f is independent of x.) Then calculus shows that for a fixed $t_0 \in I$, the general solution of the ODE (i.e., a form representing all possible solutions) is

$$x(t) = c + \int_{t_0}^{t} f(s)ds,$$

where $c \in \mathbb{F}^n$ is an arbitrary constant vector (i.e., $c_1, \ldots, c_n$ are n arbitrary constants in $\mathbb{F}$).

Provided f satisfies a Lipschitz condition (to be discussed soon), the general solution of a first-order system $x' = f(t, x)$ involves n arbitrary constants in $\mathbb{F}$ [or an arbitrary vector in $\mathbb{F}^n$] (whether or not we can express the general solution explicitly), so n scalar conditions [or one vector condition] must be given to specify a particular solution. For the example above, clearly giving $x(t_0) = x_0$ (for a known constant vector x_0) determines c, namely, $c = x_0$. In general, specifying $x(t_0) = x_0$ (these are called *initial conditions* (IC), even if t_0 is not the left endpoint of the t-interval I) determines a particular solution of the ODE.

Initial-Value Problems for First-order Systems

An initial value problem (IVP) for the first-order system is the differential equation

$$DE: \qquad x' = f(t, x),$$

together with initial conditions

$$IC: \qquad x(t_0) = x_0\ .$$

A solution to the IVP is a solution $x(t)$ of the DE defined on an interval I containing t_0, which also satisfies the IC, i.e., for which $x(t_0) = x_0$.

Examples:

(1) Let $n = 1$. The solution of the IVP:

$$\begin{aligned} DE: &\qquad x' = x^2 \\ IC: &\qquad x(1) = 1 \end{aligned}$$

is $x(t) = \frac{1}{2-t}$, which blows up as $t \to 2$. So even if f is C^∞ on all of $\mathbb{R} \times \mathbb{F}^n$, solutions of an IVP do not necessarily exist for all time t.

(2) Let $n = 1$. Consider the IVP:

$$\begin{aligned} DE: &\qquad x' = 2\sqrt{|x|} \\ IC: &\qquad x(0) = 0\ . \end{aligned}$$

For any $c \geq 0$, define $x_c(t) = 0$ for $t \leq c$ and $x_c(t) = (t-c)^2$ for $t \geq c$. Then every $x_c(t)$ for $c \geq 0$ is a solution of this IVP. So in general for continuous $f(t, x)$, the solution of an IVP might not be unique. (The difficulty here is that $f(t, x) = 2\sqrt{|x|}$ is not Lipschitz continuous near $x = 0$.)

An Integral Equation Equivalent to an IVP

Suppose $x(t) \in C^1(I)$ is a solution of the IVP:

$$\begin{aligned} DE: &\qquad x' = f(t, x) \\ IC: &\qquad x(t_0) = x_0 \end{aligned}$$

defined on an interval $I \subset \mathbb{R}$ with $t_0 \in I$. Then for all $t \in I$,

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} x'(s)ds \\ &= x_0 + \int_{t_0}^{t} f(s, x(s))ds, \end{aligned}$$

so $x(t)$ is also a solution of the *integral equation*

$$\text{(IE)} \qquad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \qquad (t \in I).$$

Conversely, suppose $x(t) \in C(I)$ is a solution of the integral equation (IE). Then $f(t, x(t)) \in C(I)$, so

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \in C^1(I)$$

and $x'(t) = f(t, x(t))$ by the Fundamental Theorem of Calculus. So x is a C^1 solution of the DE on I, and clearly $x(t_0) = x_0$, so x is a solution of the IVP. We have shown:

Proposition. On an interval I containing t_0, x is a solution of the IVP: $DE : x' = f(t, x)$; $IC : x(t_0) = x_0$ (where f is continuous) with $x \in C^1(I)$ if and only if x is a solution of the integral equation (IE) on I with $x \in C(I)$.

The integral equation (IE) is a useful way to study the IVP. We can deal with the function space of continuous functions on I without having to be concerned about differentiability: continuous solutions of (IE) are automatically C^1. Moreover, the initial condition is built into the integral equation.

We will solve (IE) using a fixed-point formulation.

Definition. Let (X, d) be a metric space, and suppose $F : X \to X$. We say that F is a *contraction* [on X] if there exists $c < 1$ such that

$$(\forall\, x, y \in X) \qquad d(F(x), F(y)) \leq c d(x, y)$$

(c is sometimes called the contraction constant). A point $x_* \in X$ for which

$$F(x_*) = x_*$$

is called a *fixed point* of F.

Theorem (Contraction Mapping Fixed-Point Theorem).
Let (X, d) be a complete *metric space and $F : X \to X$ be a contraction (with contraction constant $c < 1$). Then F has a unique fixed point $x_* \in X$. Moreover, for any $x_0 \in X$, if we generate the sequence $\{x_k\}$ iteratively by* functional iteration

$$x_{k+1} = F(x_k) \quad \textit{for} \quad k \geq 0$$

(sometimes called fixed-point iteration*), then $x_k \to x_*$.*

Proof. Fix $x_0 \in X$, and generate $\{x_k\}$ by $x_{k+1} = F(x_k)$. Then for $k \geq 1$,

$$d(x_{k+1}, x_k) = d(F(x_k), F(x_{k-1})) \leq c d(x_k, x_{k-1}).$$

By induction

$$d(x_{k+1}, x_k) \leq c^k d(x_1, x_0).$$

So for $n < m$,

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \leq \left(\sum_{j=n}^{m-1} c^j \right) d(x_1, x_0) \\ &\leq \left(\sum_{j=n}^{\infty} c^j \right) d(x_1, x_0) = \frac{c^n}{1-c} d(x_1, x_0). \end{aligned}$$

Since $c^n \to 0$ as $n \to \infty$, $\{x_k\}$ is Cauchy. Since X is complete, $x_k \to x_*$ for some $x_* \in X$. Since F is a contraction, clearly F is continuous, so

$$F(x_*) = F(\lim x_k) = \lim F(x_k) = \lim x_{k+1} = x_*,$$

so x_* is a fixed point. If x and y are two fixed points of F in X, then

$$d(x, y) = d(F(x), F(y)) \leq c d(x, y),$$

so $(1 - c) d(x, y) \leq 0$, and thus $d(x, y) = 0$ and $x = y$. So F has a unique fixed point. $\square$

Applications.

(1) *Iterative methods for linear systems* (see problem 3 on Problem Set 9).

(2) *The Inverse Function Theorem* (see problem 4 on Problem Set 9). If $\Phi : U \to \mathbb{R}^n$ is a C^1 mapping on a neighborhood $U \subset \mathbb{R}^n$ of $x_0 \in \mathbb{R}^n$ satisfying $\Phi(x_0) = y_0$ and $\Phi'(x_0) \in \mathbb{R}^{n\times n}$ is invertible, then there exist neighborhoods $U_0 \subset U$ of x_0 and V_0 of y_0 and a C^1 mapping $\Psi : V_0 \to U_0$ for which $\Phi[U_0] = V_0$ and $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity mappings on V_0 and U_0, respectively.

(In problem 4 of Problem Set 9, you will show that Φ has a continuous right inverse defined on some neighborhood of y_0. Other arguments are required to show that $\Psi \in C^1$ and that Ψ is a two-sided inverse; these are not discussed here.)

Remark. Applying the Contraction Mapping Fixed-Point Theorem (C.M.F.-P.T.) to a mapping F usually requires two steps:

(1) Construct a complete metric space X and a closed subset $S \subset X$ for which $F(S) \subset S$.

(2) Show that F is a contraction on S.

To apply the C.M.F.-P.T. to the integral equation (IE), we need a further condition on the function $f(t, x)$.

Definition. Let $I \subset \mathbb{R}$ be an interval and $\Omega \subset \mathbb{F}^n$. We say that $f(t, x)$ mapping $I \times \Omega$ into $\mathbb{F}^n$ is *uniformly Lipschitz continuous with respect to* x if there is a constant L (called the *Lipschitz constant*) for which

$$(\forall\, t \in I)(\forall\, x, y \in \Omega) \qquad |f(t, x) - f(t, y)| \le L|x - y| \ .$$

We say that f is in (C, Lip) on $I \times \Omega$ if f is continuous on $I \times \Omega$ and f is uniformly Lipschitz continuous with respect to x on $I \times \Omega$.

For simplicity, we will consider intervals $I \subset \mathbb{R}$ for which t_0 is the left endpoint. Virtually identical arguments hold if t_0 is the right endpoint of I, or if t_0 is in the interior of I (see Coddington & Levinson).

Theorem (Local Existence and Uniqueness for (IE) for Lipschitz f)
Let $I = [t_0, t_0 + \beta]$ and $\Omega = \overline{B_r(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \le r\}$, and suppose $f(t, x)$ is in (C, Lip) on $I \times \Omega$. Then there exisits $\alpha \in (0, \beta]$ for which there is a unique solution of the integral equation

$$\text{(IE)} \qquad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

in $C(I_\alpha)$, where $I_\alpha = [t_0, t_0 + \alpha]$. Moreover, we can choose α to be any positive number satisfying

$$\alpha \le \beta,\ \alpha \le \frac{r}{M}, \quad \textit{and} \quad \alpha < \frac{1}{L}, \ \textit{where} \quad M = \max_{(t,x)\in I\times\Omega} |f(t, x)|$$

and L is the Lipschitz constant for f in $I \times \Omega$.

Proof. For any $\alpha \in (0, \beta]$, let $\|\cdot\|_\infty$ denote the max-norm on $C(I_\alpha)$:

$$\text{for} \quad x \in C(I_\alpha), \quad \|x\|_\infty = \max_{t_0\le t\le t_0+\alpha} |x(t)| \ .$$

Although this norm clearly depends on α, we do not include α in the notation. Let $x_0 \in C(I_\alpha)$ denote the constant function $x_0(t) \equiv x_0$. For $\rho > 0$ let

$$X_{\alpha,\rho} = \{x \in C(I_\alpha) : \|x - x_0\|_\infty \leq \rho\}.$$

Then $X_{\alpha,\rho}$ is a complete metric space since it is a closed subset of the Banach space $(C(I_\alpha), \|\cdot\|_\infty)$. For any $\alpha \in (0, \beta]$, define $F : X_{\alpha,r} \to C(I_\alpha)$ by

$$(F(x))(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Note that F is well-defined on $X_{\alpha,r}$ and $F(x) \in C(I_\alpha)$ for $x \in X_{\alpha,r}$ since f is continuous on $I \times \overline{B_r(x_0)}$. Fixed points of F are solutions of the integral equation (IE).

Claim. Suppose $\alpha \in (0, \beta]$, $\alpha \leq \frac{r}{M}$, and $\alpha < \frac{1}{L}$. Then F maps $X_{\alpha,r}$ into itself and F is a contraction on $X_{\alpha,r}$.

Proof of Claim: If $x \in X_{\alpha,r}$, then for $t \in I_\alpha$,

$$|(F(x))(t) - x_0| \leq \int_{t_0}^{t} |f(s, x(s))|ds \leq M\alpha \leq r,$$

so $F : X_{\alpha,r} \to X_{\alpha,r}$. If $x, y \in X_{\alpha,r}$, then for $t \in I_\alpha$,

$$\begin{aligned} |(F(x))(t) - (F(y))(t)| &\leq \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))|ds \\ &\leq \int_{t_0}^{t} L|x(s) - y(s)|ds \\ &\leq L\alpha\|x - y\|_\infty, \end{aligned}$$

so

$$\|F(x) - F(y)\|_\infty \leq L\alpha\|x - y\|_\infty, \quad \text{and} \quad L\alpha < 1.$$

So by the C.M.F.-P.T., for α satisfying $0 < \alpha \leq \beta$, $\alpha \leq \frac{r}{M}$, and $\alpha < \frac{1}{L}$, F has a unique fixed point in $X_{\alpha,r}$, and thus the integral equation (IE) has a unique solution $x_*(t)$ in $X_{\alpha,r} = \{x \in C(I_\alpha) : \|x - x_0\|_\infty \leq r\}$. This is *almost* the conclusion of the Theorem, except we haven't shown x_* is the only solution in all of $C(I_\alpha)$. This uniqueness is better handled by techniques we will study soon, but we can still eke out a proof here. (Since f is only defined on $I \times \overline{B_r(x_0)}$, technically $f(t, x(t))$ does not make sense if $x \in C(I_\alpha)$ but $x \notin X_{\alpha,r}$. To make sense of the uniqueness statement for general $x \in C(I_\alpha)$, we choose some continuous extension of f to $I \times \mathbb{F}^n$.) Fix α as above. Then clearly for $0 < \gamma \leq \alpha$, $x_*|_{I_\gamma}$ is the unique fixed point of F on $X_{\gamma,r}$. Suppose $y \in C(I_\alpha)$ is a solution of (IE) on I_α (using perhaps an extension of f) with $y \not\equiv x_*$ on I_α. Let

$$\gamma_1 = \inf\{\gamma \in (0, \alpha] : y(t_0 + \gamma) \neq x_*(t_0 + \gamma)\}.$$

By continuity, $\gamma_1 < \alpha$. Since $y(t_0) = x_0$, continuity implies

$$\exists\, \gamma_0 \in (0, \alpha] \ni y|_{I_{\gamma_0}} \in X_{\gamma_0, r},$$

and thus $y(t) \equiv x_*(t)$ on I_{γ_0}. So $0 < \gamma_1 < \alpha$. Since $y(t) \equiv x_*(t)$ on I_{γ_1}, $y|_{I_{\gamma_1}} \in X_{\gamma_1, r}$. Let $\rho = M\gamma_1$; then $\rho < M\alpha \leq r$. For $t \in I_{\gamma_1}$,

$$|y(t) - x_0| = |(F(y))(t) - x_0| \leq \int_{t_0}^{t} |f(s, y(s))| ds \leq M\gamma_1 = \rho,$$

so $y|_{I_{\gamma_1}} \in X_{\gamma_1, \rho}$. By continuity, there exists $\gamma_2 \in (\gamma_1, \alpha] \ni y|_{I_{\gamma_2}} \in X_{\gamma_1, r}$. But then $y(t) \equiv x_*(t)$ on I_{γ_2}, contradicting the definition of γ_1. □

The Picard Iteration

Although hidden in a few too many details, the main idea of the proof above is to study the convergence of functional iterates of F. If we choose the initial iterate to be $x_0(t) \equiv x_0$, we obtain the classical Picard Iteration:

$$\begin{cases} x_0(t) & \equiv & x_0 \\ x_{k+1}(t) & = & x_0 + \int_{t_0}^{t} f(s, x_k(s)) ds \quad \text{for} \quad k \geq 0 \end{cases}$$

The argument in the proof of the C.M.F.-P.T. gives only *uniform* estimates of, e.g., $x_{k+1} - x_k$: $\|x_{k+1} - x_k\|_\infty \leq L\alpha \|x_k - x_{k+1}\|_\infty$, leading to the condition $\alpha < \frac{1}{L}$. For the Picard iteration (and other iterations of similar nature, e.g., for Volterra integral equations of the second kind), we can get better results using *pointwise* estimates of $x_{k+1} - x_k$. The condition $\alpha < \frac{1}{L}$ turns out to be unnecessary (we will see another way to eliminate this assumption when we study continuation of solutions). For the moment, we will set aside the uniqueness question and focus on existence.

Theorem (Picard Global Existence for (IE) for Lipschitz f). *Let $I = [t_0, t_0 + \beta]$, and suppose $f(t, x)$ is in (C, Lip) on $I \times \mathbb{F}^n$. Then there exists a solution $x_*(t)$ of the integral equation (IE) in $C(I)$.*

Theorem (Picard Local Existence for (IE) for Lipschitz f). *Let $I = [t_0, t_0 + \beta]$ and $\Omega = \overline{B_r(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \leq r\}$, and suppose $f(t, x)$ is in (C, Lip) on $I \times \Omega$. Then there exists a solution $x_*(t)$ of the integral equation (IE) in $C(I_\alpha)$, where $I_\alpha = [t_0, t_0 + \alpha]$, $\alpha = \min\left(\beta, \frac{r}{M}\right)$, and $M = \max_{(t,x) \in I \times \Omega} |f(t, x)|$.*

Proofs. We prove the two theorems together. For the global theorem, let $X = C(I)$ (i.e., $C(I, \mathbb{F}^n)$), and for the local theorem, let

$$X = X_{\alpha, r} \equiv \{x \in C(I_\alpha) : \|x - x_0\|_\infty \leq r\}$$

as before (where $x_0(t) \equiv x_0$). Then the map

$$(F(x))(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds$$

maps X into X in both cases, and X is complete. Let

$$x_0(t) \equiv x_0, \quad \text{and} \quad x_{k+1} = F(x_k) \quad \text{for} \quad k \geq 0.$$

Let

$$\begin{aligned} M_0 &= \max_{t \in I} |f(t, x_0)| \qquad \text{(global theorem)}, \\ M_0 &= \max_{t \in I_\alpha} |f(t, x_0)| \qquad \text{(local theorem)}. \end{aligned}$$

Then for $t \in I$ (global) or $t \in I_\alpha$ (local),

$$\begin{aligned} |x_1(t) - x_0| &\leq \int_{t_0}^t |f(s, x_0)| ds \leq M_0(t - t_0) \\ |x_2(t) - x_1(t)| &\leq \int_{t_0}^t |f(s, x_1(s)) - f(s, x_0(s))| ds \\ &\leq L \int_{t_0}^t |x_1(s) - x_0(s)| ds \\ &\leq M_0 L \int_{t_0}^t (s - t_0) ds = \frac{M_0 L (t - t_0)^2}{2!} \end{aligned}$$

By induction, suppose $|x_k(t) - x_{k-1}(t)| \leq M_0 L^{k-1} \frac{(t-t_0)^k}{k!}$. Then

$$\begin{aligned} |x_{k+1}(t) - x_k(t)| &\leq \int_{t_0}^t |f(s, x_k(s)) - f(s, x_{k-1}(s))| ds \\ &\leq L \int_{t_0}^t |x_k(s) - x_{k-1}(s)| ds \\ &\leq M_0 L^k \int_{t_0}^t \frac{(s - t_0)^k}{k!} ds = M_0 L^k \frac{(t - t_0)^{k+1}}{(k+1)!}. \end{aligned}$$

So

$$\begin{aligned} \sum_{k=0}^{\infty} |x_{k+1}(t) - x_k(t)| &\leq \frac{M_0}{L} \sum_{k=0}^{\infty} \frac{(L(t - t_0))^{k+1}}{(k+1)!} \\ &= \frac{M_0}{L}(e^{L(t-t_0)} - 1) \\ &\leq \frac{M_0}{L}(e^{L\gamma} - 1) \end{aligned}$$

where $\gamma = \beta$ (global) or $\gamma = \alpha$ (local). Hence the series $x_0 + \sum_{k=0}^{\infty}(x_{k+1}(t) - x_k(t))$, which has x_{N+1} as its N^{th} partial sum, converges absolutely and uniformly on I (global) or I_α (local) by the Weierstrass M-test. Let $x_*(t) \in C(I)$ (global) or $\in C(I_\alpha)$ (local) be the limit function. Since

$$|f(t, x_k(t)) - f(t, x_*(t))| \leq L |x_k(t) - x_*(t)|,$$

$f(t, x_k(t))$ converges uniformly to $f(t, x_*(t))$ on I (global) or I_α (local), and thus

$$\begin{aligned} F(x_*)(t) &= x_0 + \int_{t_0}^{t} f(s, x_*(s))ds \\ &= \lim_{k\to\infty} (x_0 + \int_{t_0}^{t} f(s, x_k(x))ds) \\ &= \lim_{k\to\infty} x_{k+1}(t) = x_*(t), \end{aligned}$$

for all $t \in I$ (global) or I_α (local). Hence $x_*(t)$ is a fixed point of F in X, and thus also a solution of the integral equation (IE) in $C(I)$ (global) or $C(I_\alpha)$ (local). □

Corollary. The solution $x_*(t)$ of (IE) satisfies

$$|x_*(t) - x_0| \le \frac{M_0}{L}(e^{L(t-t_0)} - 1)$$

for $t \in I$ (global) or $t \in I_\alpha$ (local), where $M_0 = \max_{t\in I} |f(t, x_0)|$ (global), or $M_0 = \max_{t\in I_\alpha} |f(t, x_0)|$ (local).

Proof. This is established in the proof above. □

Remark. In each of the statements of the last three Theorems, we could replace "solution of the integral equation (IE)" with "solution of the IVP: $DE : x' = f(t, x)$; $IC : x(t_0) = x_0$" because of the equivalence of these two problems.

Examples.

(1) Consider a *linear* system $x' = A(t)x + b(t)$, where $A(t) \in \mathbb{C}^{n\times n}$ and $b(t) \in \mathbb{C}^n$ are in $C(I)$ (where $I = [t_0, t_0 + \beta]$). Then f is in (C, Lip) on $I \times \mathbb{F}^n$:

$$|f(t, x) - f(t, y)| \le |A(t)x - A(t)y| \le \left(\max_{t\in I} \|A(t)\|\right) |x - y|.$$

Hence there is a solution of the IVP: $x' = A(t)x + b(t)$, $x(t_0) = x_0$ in $C^1(I)$.

(2) $(n = 1)$ Consider the IVP: $x' = x^2$, $x(0) = x_0 > 0$. Then $f(t, x) = x^2$ is not in (C, Lip) on $I \times \mathbb{R}$. It is, however, in (C, Lip) on $I \times \Omega$ where $\Omega = \overline{B_r(x_0)} = [x_0 - r, x_0 + r]$ for each fixed r. For a given $r > 0$, $M = (x_0 + r)^2$, and $\alpha = \frac{r}{M} = \frac{r}{(x_0+r)^2}$ in the local theorem is maximized for $r = x_0$, for which $\alpha = (4x_0)^{-1}$. So the local theorem guarantees a solution in $[0, (4x_0)^{-1}]$. The actual solution $x_*(t) = (x_0^{-1} - t)^{-1}$ exists in $[0, (x_0)^{-1})$.

Local Existence for Continuous f

Some condition similar to the Lipschitz condition is needed to guarantee that the Picard iterates converge; it is also needed for uniqueness, which we will return to shortly. It is, however, still possible to prove a local existence theorem assuming only that f is continuous, without assuming the Lipschitz condition. We will need the following form of Ascoli's Theorem:

Theorem (Ascoli). *Let X and Y be metric spaces with X compact. Let $\{f_k\}$ be an* equicontinuous *sequence of functions $f_k : X \to Y$, i.e.,*

$$(\forall\, \epsilon > 0)(\exists\, \delta > 0) \quad \text{such that} \quad (\forall\, k \geq 1)(\forall\, x_1, x_2 \in X)$$
$$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f_k(x_1), f_k(x_2)) < \epsilon$$

(in particular, each f_k is continuous), and suppose for each $x \in X$, $\overline{\{f_k(x) : k \geq 1\}}$ is a compact subset of Y. Then there is a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ and a continuous $f : X \to Y$ such that

$$f_{k_j} \to f \quad \text{uniformly on } X.$$

Remark. If $Y = \mathbb{F}^n$, the condition $(\forall\, x \in X)$ $\overline{\{f_k(x) : k \geq 1\}}$ is compact is equivalent to the sequence $\{f_k\}$ being *pointwise bounded*, i.e.,

$$(\forall\, x \in X)(\exists\, M_x) \quad \text{such that} \quad (\forall\, k \geq 1) \quad |f_k(x)| \leq M_x.$$

Example. Suppose $f_k : [a, b] \to \mathbb{R}$ is a sequence of C^1 functions, and suppose there exists $M > 0$ such that

$$(\forall\, k \geq 1) \quad \|f_k\|_\infty + \|f_k'\|_\infty \leq M$$

(where $\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|$). Then for $a \leq x_1 < x_2 \leq b$,

$$|f_k(x_2) - f_k(x_1)| \leq \int_{x_1}^{x_2} |f_k'(x)| dx \leq M|x_2 - x_1|,$$

so $\{f_k\}$ is equicontinuous (take $\delta = \frac{\epsilon}{M}$), and $\|f_k\|_\infty \leq M$ certainly implies $\{f_k\}$ is pointwise bounded. So by Ascoli's Theorem, some subsequence of $\{f_k\}$ converges uniformly to a continuous function $f : [a, b] \to \mathbb{R}$.

Theorem (Cauchy-Peano Existence Theorem).
Let $I = [t_0, t_0 + \beta]$ and $\Omega = \overline{B_r(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \leq r\}$, and suppose $f(t, x)$ is continuous on $I \times \Omega$. Then there exists a solution $x_*(t)$ of the integral equation

$$\text{(IE)} \qquad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds$$

in $C(I_\alpha)$ where $I_\alpha = [t_0, t_0 + \alpha]$, $\alpha = \min\left(\beta, \frac{r}{M}\right)$, and $M = \max_{(t,x) \in I \times \Omega} |f(t, x)|$ (and thus $x_*(t)$ is a C^1 solution on I_α of the IVP: $x' = f(t, x)$; $x(t_0) = x_0$).

Proof. The idea of the proof is to construct continuous approximate solutions explicitly (we will use the piecewise linear interpolants of grid functions generated by Euler's method), and use Ascoli's Theorem to take the uniform limit of some subsequence. For each integer $k \geq 1$, define $x_k(t) \in C(I_\alpha)$ as follows: partition $[t_0, t_0+\alpha]$ into k equal subintervals (for $0 \leq \ell \leq k$, let $t_\ell = t_0 + \ell\frac{\alpha}{k}$ (note: t_ℓ depends on k too)), set $x_k(t_0) = x_0$, and for $\ell = 1, 2, \ldots, k$ define $x_k(t)$ in $(t_{\ell-1}, t_\ell]$ inductively by $x_k(t) = x_k(t_{\ell-1}) + f(t_{\ell-1}, x_k(t_{\ell-1}))(t - t_{\ell-1})$. For this to be well-defined we must check that $|x_k(t_{\ell-1}) - x_0| \leq r$ for $2 \leq \ell \leq k$ (it is obvious for $\ell = 1$); inductively, we have

$$\begin{aligned}
|x_k(t_{\ell-1}) - x_0| &\leq \sum_{i=1}^{\ell-1} |x_k(t_i) - x_k(t_{i-1})| \\
&= \sum_{i=1}^{\ell-1} |f(t_{i-1}, x_k(t_{i-1}))| \cdot |t_i - t_{i-1}| \\
&\leq M \sum_{i=1}^{\ell-1} (t_i - t_{i-1}) \\
&= M(t_{\ell-1} - t_0) \leq M\alpha \leq r
\end{aligned}$$

by the choice of α. So $x_k(t) \in C(I_\alpha)$ is well defined. A similar estimate shows that for $t, \tau \in [t_0, t_0+\alpha]$,

$$|x_k(t) - x_k(\tau)| \leq M|t - \tau|.$$

This implies that $\{x_k\}$ is equicontinuous; it also implies that

$$(\forall\, k \geq 1)(\forall\, t \in I_\alpha) \quad |x_k(t) - x_0| \leq M\alpha \leq r,$$

so $\{x_k\}$ is pointwise bounded (in fact, uniformly bounded). So by Ascoli's Theorem, there exists $x_*(t) \in C(I_\alpha)$ and a subsequence $\{x_{k_j}\}_{j=1}^\infty$ converging uniformly to $x_*(t)$. It remains to show that $x_*(t)$ is a solution of (IE) on I_α. Since each $x_k(t)$ is continuous and piecewise linear on I_α,

$$x_k(t) = x_0 + \int_{t_0}^t x_k'(s)ds$$

(where $x_k'(t)$ is piecewise constant on I_α and is defined for all t except t_ℓ $(1 \leq \ell \leq k-1)$, where we define it to be $x_k'(t_\ell^+)$). Define

$$\Delta_k(t) = x_k'(t) - f(t, x_k(t)) \quad \text{on} \quad I_\alpha$$

(note that $\Delta_k(t_\ell) = 0$ for $0 \leq \ell \leq k-1$ by definition). We claim that $\Delta_k(t) \to 0$ uniformly on I_α as $k \to \infty$. Indeed, given k, we have for $1 \leq \ell \leq k$ and $t \in (t_{\ell-1}, t_\ell)$ (including t_k if $\ell = k$), that

$$|x_k'(t) - f(t, x_k(t))| = |f(t_{\ell-1}, x_k(t_{\ell-1})) - f(t, x_k(t))|.$$

Noting that $|t - t_{\ell-1}| \leq \frac{\alpha}{k}$ and

$$|x_k(t) - x_k(t_{\ell-1})| \leq M|t - t_{\ell-1}| \leq M\frac{\alpha}{k},$$

the uniform continuity of f (being continuous on the compact set $I \times \Omega$) implies that

$$\max_{t \in I_\alpha} |\Delta_k(t)| \to 0 \quad \text{as} \quad k \to \infty.$$

Thus, in particular, $\Delta_{k_j}(t) \to 0$ uniformly on I_α. Now

$$\begin{aligned} x_{k_j}(t) &= x_0 + \int_{t_0}^t x'_{k_j}(s)ds \\ &= x_0 + \int_{t_0}^t f(s, x_{k_j}(s))ds + \int_{t_0}^t \Delta_{k_j}(s)ds. \end{aligned}$$

Since $x_{k_j} \to x_*$ uniformly on I_α, the uniform continuity of f on $I \times \Omega$ now implies that $f(t, x_{k_j}(t)) \to f(t, x_*(t))$ uniformly on I_α, so taking the limit as $j \to \infty$ on both sides of this equation for each $t \in I_\alpha$, we obtain that x_* satisfies (IE) on I_α. $\square$

Remark. In general, the choice of a subsequence of $\{x_k\}$ is necessary: there are examples where the sequence $\{x_k\}$ does not converge. (See Problem 12, Chapter 1 of Coddington & Levinson.)

Uniqueness

Uniqueness theorems are typically proved by comparison theorems for solutions of scalar differential equations, or by inequalities. The most fundamental of these inequalities is Gronwall's inequality, which applies to real first-order linear scalar equations.

Recall that a first-order linear scalar initial value problem

$$u' = a(t)u + b(t), \quad u(t_0) = u_0$$

can be solved by multiplying by the integrating factor $e^{-\int_{t_0}^t a}$ (i.e., $e^{-\int_{t_0}^t a(s)ds}$), and then integrating from t_0 to t. That is,

$$\frac{d}{dt}\left(e^{-\int_{t_0}^t a} u(t)\right) = e^{-\int_{t_0}^t a} b(t),$$

which implies that

$$\begin{aligned} e^{-\int_{t_0}^t a} u(t) - u_0 &= \int_{t_0}^t \frac{d}{ds}\left(e^{-\int_{t_0}^s a} u(s)\right) ds \\ &= \int_{t_0}^t e^{-\int_{t_0}^s a} b(s)ds \end{aligned}$$

which in turn implies that

$$u(t) = u_0 e^{\int_{t_0}^t a} + \int_{t_0}^t e^{\int_s^t a} b(s)ds.$$

Since $f(t) \le g(t)$ on $[c, d]$ implies $\int_c^d f(t)dt \le \int_c^d g(t)dt$, the identical argument with "$=$" replaced by "$\le$" gives

Theorem (Gronwall's Inequality - differential form). *Let $I = [t_0, t_1]$. Suppose $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ are continuous, and suppose $u : I \to \mathbb{R}$ is in $C^1(I)$ and satisfies*

$$u'(t) \le a(t)u(t) + b(t) \quad \text{for} \quad t \in I, \quad \text{and} \quad u(t_0) = u_0.$$

Then

$$u(t) \le u_0 e^{\int_{t_0}^t a} + \int_{t_0}^t e^{\int_s^t a} b(s)ds.$$

Remarks:

(1) Thus a solution of the differential inequality is bounded above by the solution of the equality (i.e., the differential equation $u' = au + b$).

(2) The result clearly still holds if u is only continuous and piecewise C^1, and $a(t)$ and $b(t)$ are only piecewise continuous.

(3) There is also an integral form of Gronwall's inequality (i.e., the hypothesis is an integral inequality): if $\varphi, \psi, \alpha \in C(I)$ are real-valued with $\alpha \ge 0$ on I, and

$$\varphi(t) \le \psi(t) + \int_{t_0}^t \alpha(s)\varphi(s)ds \quad \text{for} \quad t \in I,$$

then

$$\varphi(t) \le \psi(t) + \int_{t_0}^t e^{\int_s^t \alpha} \alpha(s)\psi(s)ds.$$

In particular, if $\psi(t) \equiv c$ (a constant), then $\varphi(t) \le c e^{\int_{t_0}^t \alpha}$. (The differential form is applied to the C^1 function $u(t) = \int_{t_0}^t \alpha(s)\varphi(s)ds$ in the proof.)

(4) For $a(t) \ge 0$, the differential form is also a consequence of the integral form: integrating

$$u' \le a(t)u + b(t)$$

from t_0 to t gives

$$u(t) \le \psi(t) + \int_{t_0}^t a(s)u(s)ds,$$

where

$$\psi(t) = u_0 + \int_{t_0}^t b(s)ds,$$

so the integral form and then integration by parts give

$$\begin{aligned} u(t) &\le \psi(t) + \int_{t_0}^t e^{\int_s^t a} a(s)\psi(s)ds \\ &= \cdots = u_0 e^{\int_{t_0}^t a} + \int_{t_0}^t e^{\int_s^t a} b(s)ds. \end{aligned}$$

(5) Caution: a differential inequality implies an integral inequality, but *not* vice versa: $f \le g \not\Rightarrow f' \le g'$.

(6) The integral form doesn't require $\varphi \in C^1$ (just $\varphi \in C(I)$), but is restricted to $\alpha \ge 0$. The differential form has no sign restriction on $a(t)$, but it requires a stronger hypothesis (in view of (5) and the requirement that u be continuous and piecewise C^1).

Uniqueness for Locally Lipschitz f

We start with a one-sided local uniqueness theorem for the initial value problem

$$IVP: \qquad x' = f(t,x); \quad x(t_0) = x_0.$$

Theorem. *Suppose for some $\alpha > 0$ and some $r > 0$, $f(t,x)$ is in (C, Lip) on $I_\alpha \times \overline{B_r(x_0)}$, and suppose $x(t)$ and $y(t)$ both map I_α into $\overline{B_r(x_0)}$ and both are C^1 solutions of (IVP) on I_α, where $I_\alpha = [t_0, t_0 + \alpha]$. Then $x(t) = y(t)$ for $t \in I_\alpha$.*

Proof. Set

$$u(t) = |x(t) - y(t)|^2 = \langle x(t) - y(t), x(t) - y(t) \rangle$$

(in the Euclidean inner product on $\mathbb{F}^n$). Then $u : I_\alpha \to [0, \infty)$ and $u \in C^1(I_\alpha)$ and for $t \in I_\alpha$,

$$\begin{aligned} u' &= \langle x - y, x' - y' \rangle + \langle x' - y', x - y \rangle \\ &= 2\mathcal{R}e\langle x - y, x' - y' \rangle \leq 2|\langle x - y, x' - y' \rangle| \\ &= 2|\langle x - y, (f(t,x) - f(t,y)) \rangle| \\ &\leq 2|x - y| \cdot |f(t,x) - f(t,y)| \\ &\leq 2L|x - y|^2 = 2Lu \ . \end{aligned}$$

Thus $u' \leq 2Lu$ on I_α and $u(t_0) = x(t_0) - y(t_0) = x_0 - x_0 = 0$. By Gronwall's inequality, $u(t) \leq u_0 e^{2Lt} = 0$ on I_α, so since $u(t) \geq 0$, $u(t) \equiv 0$ on I_α. □

Corollary.

(i) The same result holds if $I_\alpha = [t_0 - \alpha, t_0]$.

(ii) The same result holds if $I_\alpha = [t_0 - \alpha, t_0 + \alpha]$.

Proof. For (i), let $\widetilde{x}(t) = x(2t_0 - t)$, $\widetilde{y}(t) = y(2t_0 - t)$, and $\widetilde{f}(t,x) = -f(2t_0 - t, x)$. Then $\widetilde{f}$ is in (C, Lip) on $[t_0, t_0 + \alpha] \times \overline{B_r(x_0)}$, and $\widetilde{x}$ and $\widetilde{y}$ both satisfy the IVP

$$x' = \widetilde{f}(t,x); \quad x'(t_0) = x_0 \quad \text{on} \quad [t_0, t_0 + \alpha].$$

So by the Theorem, $\widetilde{x}(t) = \widetilde{y}(t)$ for $t \in [t_0, t_0 + \alpha]$, i.e., $x(t) = y(t)$ for $t \in [t_0 - \alpha, t_0]$. Now (ii) follows immediately by applying the Theorem in $[t_0, t_0 + \alpha]$ and applying (i) in $[t_0 - \alpha, t_0]$. □

Remark. The idea used in the proof of (i) is often called "time-reversal." The important part is that $\widetilde{x}(t) = x(c - t)$, etc., for some constant c, so that $\widetilde{x}'(t) = -x'(c - t)$, etc. The choice of $c = 2t_0$ is convenient but not essential.

The main uniqueness theorem is easiest to formulate in the case when the initial point (t_0, x_0) is in the interior of the domain of definition of f. There are analogous results with essentially the same proof when (t_0, x_0) is on the boundary of the domain of definition of f.

(Exercise: State precisely a theorem corresponding to the upcoming theorem which applies in such a situation.)

Definition. Let $\mathcal{D}$ be an open set in $\mathbb{R} \times \mathbb{F}^n$. We say that $f(t,x)$ mapping $\mathcal{D}$ into $\mathbb{F}^n$ is *locally Lipschitz continuous with respect to* x if for each $(t_1, x_1) \in \mathcal{D}$ there exists

$$\alpha > 0, \quad r > 0, \quad \text{and} \quad L > 0$$

for which $[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)} \subset \mathcal{D}$ and

$$(\forall\, t \in [t_1 - \alpha, t_1 + \alpha])(\forall\, x, y \in \overline{B_r(x_1)}) \quad |f(t,x) - f(t,y)| \leq L|x - y|$$

(i.e., f is uniformly Lipschitz continuous with respect to x in $[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)}$). We will say $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ (not a standard notation) on $\mathcal{D}$ if f is continuous on $\mathcal{D}$ and locally Lipschitz continuous with respect to x on $\mathcal{D}$.

Example. Let $\mathcal{D}$ be an open set of $\mathbb{R} \times \mathbb{F}^n$. Suppose $f(t,x)$ maps $\mathcal{D}$ into $\mathbb{F}^n$, f is continuous on $\mathcal{D}$, and for $1 \leq i, j \leq n$, $\frac{\partial f_i}{\partial x_j}$ exists and is continuous in $\mathcal{D}$. (Briefly, we say f is continuous on $\mathcal{D}$ and C^1 with respect to x on $\mathcal{D}$.) Then $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on $\mathcal{D}$. (Exercise.)

Main Uniqueness Theorem. *Let $\mathcal{D}$ be an open set in $\mathbb{R} \times \mathbb{F}^n$, and suppose $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on $\mathcal{D}$. Suppose $(t_0, x_0) \in \mathcal{D}$, $I \subset \mathbb{R}$ is some interval containing t_0 (which may be open or closed at either end), and suppose $x(t)$ and $y(t)$ are both solutions of the initial value problem*

$$IVP: \qquad x' = f(t,x); \quad x(t_0) = x_0$$

in $C^1(I)$. (Included in this hypothesis is the assumption that $(t, x(t)) \in \mathcal{D}$ and $(t, y(t)) \in \mathcal{D}$ for $t \in I$.) Then $x(t) \equiv y(t)$ on I.

Proof. We first show $x(t) \equiv y(t)$ on $\{t \in I : t \geq t_0\}$. If not, let

$$t_1 = \inf\{t \in I : t \geq t_0 \quad \text{and} \quad x(t) \neq y(t)\}.$$

Then $x(t) = y(t)$ on $[t_0, t_1)$ so by continuity $x(t_1) = y(t_1)$ (if $t_1 = t_0$, this is obvious). Set $x_1 = x(t_1) = y(t_1)$. By continuity and the openness of $\mathcal{D}$ (as $(t_1, x_1) \in \mathcal{D}$), there exist $\alpha > 0$ and $r > 0$ such that $[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)} \subset \mathcal{D}$, f is uniformly Lipschitz continuous with respect to x in $[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)}$, and $x(t) \in \overline{B_r(x_1)}$ and $y(t) \in \overline{B_r(x_1)}$ for all t in $I \cap [t_1 - \alpha, t_1 + \alpha]$. By the previous theorem, $x(t) \equiv y(t)$ in $I \cap [t_1 - \alpha, t_1 + \alpha]$, contradicting the definition of t_1. Hence $x(t) \equiv y(t)$ on $\{t \in I : t \geq t_0\}$. Similarly, $x(t) \equiv y(t)$ on $\{t \in I : t \leq t_0\}$. Hence $x(t) \equiv y(t)$ on I. $\square$

Remark. t_0 is allowed to be the left or right endpoint of I.

Comparison Theorem for Nonlinear Real Scalar Equations

We conclude this section with a version of Gronwall's inequality for nonlinear equations.

Theorem. *Let $n = 1$, $\mathbb{F} = \mathbb{R}$. Suppose $f(t,u)$ is continuous in t and u and Lipschitz continuous in u. Suppose $u(t)$, $v(t)$ are C^1 for $t \geq t_0$ (or some interval $[t_0, b)$ or $[t_0, b]$) and satisfy*

$$u'(t) \leq f(t, u(t)), \qquad v'(t) = f(t, v(t))$$

and $u(t_0) \le v(t_0)$. Then $u(t) \le v(t)$ for $t \ge t_0$.

Proof. By contradiction. If $u(T) > v(T)$ for some $T > t_0$, then set

$$t_1 = \sup\{t : t_0 \le t < T \quad \text{and} \quad u(t) \le v(t)\}.$$

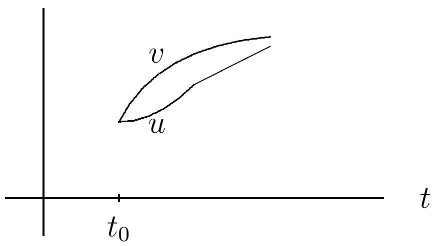
Then $t_0 \le t_1 < T$, $u(t_1) = v(t_1)$, and $u(t) > v(t)$ for $t > t_1$ (using continuity of $u - v$). For $t_1 \le t \le T$, $|u(t) - v(t)| = u(t) - v(t)$, so we have

$$(u - v)' \le f(t, u) - f(t, v) \le L|u - v| = L(u - v).$$

By Gronwall's inequality (applied to $u-v$ on $[t_1, T]$, with $(u-v)(t_1) = 0$, $a(t) \equiv L$, $b(t) \equiv 0$), $(u - v)(t) \le 0$ on $[t_1, T]$, a contradiction. □

Remarks.

1. As with the differential form of Gronwall's inequality, a solution of the differential inequality $u' \le f(t, u)$ is bounded above by the solution of the equality (i.e., the DE $v' = f(t, v)$).
2. It can be shown under the same hypotheses that if $u(t_0) < v(t_0)$, then $u(t) < v(t)$ for $t \ge t_0$ (problem 4 on Problem Set 1).
3. Caution: It may happen that $u'(t) > v'(t)$ for some $t \ge t_0$. It is not true that $u(t) \le v(t) \Rightarrow u'(t) \le v'(t)$, as illustrated in the picture below.



Corollary. Let $n = 1$, $\mathbb{F} = \mathbb{R}$. Suppose $f(t, u) \le g(t, u)$ are continuous in t and u, and one of them is Lipschitz continuous in u. Suppose also that $u(t)$, $v(t)$ are C^1 for $t \ge t_0$ (or some interval $[t_0, b)$ or $[t_0, b]$) and satisfy $u' = f(t, u)$, $v' = g(t, v)$, and $u(t_0) \le v(t_0)$. Then $u(t) \le v(t)$ for $t \ge t_0$.

Proof. Suppose first that g satisfies the Lipschitz condition. Then $u' = f(t, u) \le g(t, u)$. Now apply the theorem. If f satisfies the Lipschitz condition, apply the first part of this proof to $\widetilde{u}(t) \equiv -v(t)$, $\widetilde{v}(t) \equiv -u(t)$, $\widetilde{f}(t, u) = -g(t, -u)$, $\widetilde{g}(t, u) = -f(t, -u)$. □

Remark. Again, if $u(t_0) < v(t_0)$, then $u(t) < v(t)$ for $t \ge t_0$.

Continuation of Solutions

We consider two kinds of results:

- local continuation (continuation at a point — no Lipschitz condition assumed)
- global continuation (for locally Lipschitz f)

Continuation at a Point

Suppose $x(t)$ is a solution of the DE $x' = f(t, x)$ on an interval I and that f is continuous on some subset $\mathcal{S} \subset \mathbb{R} \times \mathbb{F}^n$ containing $\{(t, x(t)) : t \in I\}$. (Note: No Lipschitz condition is assumed.)

Case 1. I is closed at the right end, i.e., $I = (-\infty, b]$, $[a, b]$, or $(a, b]$.
Assume further that $(b, x(b))$ is in the interior of $\mathcal{S}$. Then the solution can be extended (by the Cauchy-Peano Existence Theorem) to an interval with right end $b + \beta$ for some $\beta > 0$. (Solve the IVP $x' = f(t, x)$ with initial value $x(b)$ at $t = b$ on some interval $[b, b + \beta]$ by Cauchy-Peano. To show that the connection is C^1 at $t = b$, note that the extended $x(t)$ satisfies the integral equation $x(t) = x(b) + \int_b^t f(s, x(s))ds$ on the extended interval $I \cup [b, b + \beta]$.)

Case 2. I is open at the right end, i.e., $I = (-\infty, b)$, $[a, b)$, or (a, b) with $b < \infty$.
Assume further that $f(t, x(t))$ is *bounded* on $[t_0, b)$ for some $t_0 < b$ with $[t_0, b) \subset I$, say $|f(t, x(t))| \le M$ on $[t_0, b)$.

Remarks about this assumption:

1. If this is true for some $\widetilde{t_0} \in I$, it is true for all $t_0 \in I$ (where of course M depends on t_0): for $t_0 < \widetilde{t_0}$, $f(t, x(t))$ is continuous on $[t_0, \widetilde{t_0}]$. So the assumption is a condition on the behavior of $f(t, x(t))$ near $t = b$.
2. The assumption can be restated with a slightly different emphasis: for some $t_0 \in I$, $\{(t, x(t)) : t_0 \le t < b\}$ stays within a subset of $\mathcal{S}$ on which f is bounded. For example, if $\{(t, x(t)) : t_0 \le t < b\}$ stays within a compact subset of $\mathcal{S}$, this condition is satisfied.

The integral equation

$$\text{(IE)} \qquad x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds$$

holds for $t \in I$. In particular, for $t_0 \le \tau \le t < b$,

$$|x(t) - x(\tau)| = \left| \int_\tau^t f(s, x(s))ds \right| \le \int_\tau^t |f(s, x(s))|ds \le M|t - \tau|.$$

Thus, for any sequence $t_n \uparrow b$, $\{x(t_n)\}$ is Cauchy. This implies $\lim_{t \to b^-} x(t)$ exists; call it $x(b^-)$. So $x(t)$ has a *continuous* extension from I to $I \cup \{b\}$. If in addition $(b, x(b^-))$ is in

$\mathcal{S}$, then (IE) holds on $I \cup \{b\}$ as well, so $x(t)$ is a C^1 *solution* of $x' = f(t,x)$ on $I \cup \{b\}$. (Of course, if now in addition $(b, x(b^-))$ is in the interior of $\mathcal{S}$, we are back in Case 1 and can extend the solution $x(t)$ a little beyond $t = b$.)

Case 3. I is closed at the left end — similar to Case 1.

Case 4. I is open at the left end — similar to Case 2.

Global Continuation

Now suppose $f(t,x)$ is continuous on an open set $\mathcal{D} \subset \mathbb{R} \times \mathbb{F}^n$ and suppose f is locally Lipschitz continuous with respect to x on $\mathcal{D}$. (For example, if f is C^1 with respect to x in $\mathcal{D}$, i.e., $\frac{\partial f_i}{\partial x_j}$ exists and is continuous in $\mathcal{D}$ for $1 \le i, j \le n$, then f is locally Lipschitz continuous with respect to x on $\mathcal{D}$.) For brevity, we will write $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on $\mathcal{D}$. Let $(t_0, x_0) \in \mathcal{D}$. We want to continue in t solutions of the IVP $x' = f(t,x)$, $x(t_0) = x_0$. Part of being a solution is that $(t, x(t)) \in \mathcal{D}$ (we are only assuming f is defined in $\mathcal{D}$). We know local existence of solutions and uniqueness of solutions on any interval.

Define

$$T_+ = \sup\{t > t_0 : \text{ there exists a solution of the IVP on } [t_0, t)\}.$$

By uniqueness, two solutions must agree on their common interval of definition, so there exists a solution on $[t_0, T_+)$. Define T_- similarly. So (T_-, T_+) is the maximal interval of existence of the solution of the IVP. It is possible that $T_+ = \infty$ and/or $T_- = -\infty$. Note that the maximal interval (T_-, T_+) is open: if the solution could be extended to T_+ (or T_-), then since $\mathcal{D}$ is open, the results above on continuation at a point imply that the solution could be extended beyond T_+ (or T_-), contradicting the definition of T_+ (or T_-).

The ideal situation would be $T_+ = +\infty$ and $T_- = -\infty$, in which case the solution exists for all time t. Another "good" situation is if $f(t,x)$ is not defined for $t \ge T_+$. For example, if $a(t) = \frac{1}{1-t}$ (which blows up at $t = 1$), and $x'(t) = a(t)$, we don't expect the solution to exist beyond $t = 1$. Here, if $t_0 = 0$ and $\mathcal{D} = (-\infty, 1) \times \mathbb{R}$, then $T_+ = 1$.

Other less desirable behavior occurs for $x' = x^2$, $x(0) = x_0 > 0$, $t_0 = 0$, and $\mathcal{D} = \mathbb{R} \times \mathbb{R}$. The solution $x(t) = x_0(1 - x_0 t)^{-1}$ blows up at time $T_+ = x_0^{-1}$ (note that $T_- = -\infty$). Observe that $x(t) \to \infty$ as $t \to (T_+)^-$. So the solution does not just "stop" in the interior of $\mathcal{D}$. This is the general behavior in this situation.

Theorem. Suppose $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on an open set $\mathcal{D} \subset \mathbb{R} \times \mathbb{F}^n$. Let $(t_0, x_0) \in \mathcal{D}$, and let (T_-, T_+) be the maximal interval of existence of the solution of the IVP $x' = f(t,x)$, $x(t_0) = x_0$. Given a compact set $K \subset \mathcal{D}$, there exists a $T < T_+$ for which $(t, x(t)) \notin K$ for $t > T$.

Proof. If not, there exists $t_j \to T_+$ with $(t_j, x(t_j)) \in K$ for all j. By taking a subsequence, we may assume that $x(t_j)$ also converges, say to $x_+ \in \mathbb{F}^n$, and $(t_j, x(t_j)) \to (T_+, x_+) \in K \subset \mathcal{D}$. We can thus choose r, τ, N such that

$$\bigcup_{j=N}^{\infty} \{(t,x) : |t - t_j| \le \tau, |x - x(t_j)| \le r\}$$

is contained in a compact subset of $\mathcal{D}$. There is an M for which $|f(t,x)| \le M$ on this compact set. By the local existence theorem, the solution of $x' = f(t,x)$ starting at the initial point $(t_j, x(t_j))$ exists for a time interval of length $T' \equiv \min\left\{\tau, \frac{r}{M}\right\}$, *independent of* j. Choose j for which $t_j > T_+ - T'$. Then the solution $x(t)$ to IVP exists beyond time T_+, which is a contradiction. $\square$

Continuation for Autonomous Systems

The system of ODE's $x'(t) = f(t,x)$ is called an *autonomous system* if $f(t,x)$ is independent of t, i.e., the ODE is of the form $x' = f(x)$.

Remarks.

(1) Time translates of solutions of an autonomous system are again solutions: if $x(t)$ is a solution, so is $x(t-c)$ for constant c.

(2) Any system of ODE's $x' = f(t,x)$ is equivalent to an autonomous system. Define "$x_0 = t$" as follows. Set $\widetilde{x} = (x_0, x) \in \mathbb{F}^{n+1}$ and define

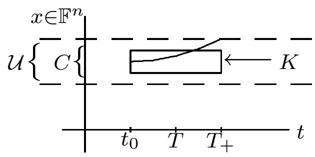
$$\widetilde{f}(\widetilde{x}) = \widetilde{f}(x_0, x) = \begin{bmatrix} 1 \\ f(x_0, x) \end{bmatrix} \in \mathbb{F}^{n+1}.$$

Then the autonomous IVP

$$\widetilde{x}' = \widetilde{f}(\widetilde{x}), \qquad \widetilde{x}(t_0) = \begin{bmatrix} t_0 \\ x_0 \end{bmatrix}$$

is equivalent to the IVP $x' = f(t,x)$, $x(t_0) = x_0$.

Consider the theorem of the previous section in the case of an autonomous IVP $x' = f(x)$, $x(t_0) = x_0$. Suppose $f(x)$ is defined and locally Lipschitz continuous on an open set $\mathcal{U} \subset \mathbb{F}^n$ and $x_0 \in \mathcal{U}$. Take $\mathcal{D} = \mathbb{R} \times \mathcal{U}$. Suppose $T_+ < \infty$ and C is a compact subset of $\mathcal{U}$. Take $K = [t_0, T_+] \times C$ in the previous theorem. The picture in $\mathbb{R} \times \mathbb{F}^n$ is:

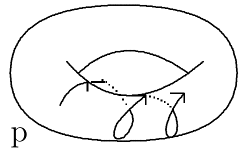


The continuation theorem implies that there exists $T < T_+$ for which $x(t) \notin C$ for $T < t < T_+$. Thus for every $C^{\text{compact}} \subset \mathcal{U}$, there exists $T < T_+$ such that $x(t) \notin C$ for $t \in (T, T_+)$. Stated briefly, eventually $x(t)$ stays out of any given compact set. This conclusion is sometimes stated informally as $x(t) \to \partial\mathcal{U} \cup \{\infty\}$ as $t \to (T_+)^-$.

The contrapositive of this statement is that if $x(t)$ stays in a compact set $C \subset \mathcal{U}$ for all $t < T_+$, then $T_+ = \infty$. Thus one can conclude that a solution exists for all time if one can show that it stays in some compact set.

In case $\mathbb{F} = \mathbb{R}$, for $\mathcal{U} \subset \mathbb{R}^n$ one can interpret a function $f : \mathcal{U} \to \mathbb{R}^n$ as a vector field on $\mathcal{U}$. The geometric interpretation of the differential equation $x' = f(x)$ is that the curve $t \to x(t)$ is an integral curve of the vector field f; i.e., for each t, the tangent vector to the curve at the point $x(t)$ is $f(x(t))$.

Example. As an application of the continuation theorem, suppose there is a smooth compact hypersurface $\mathcal{S} \subset \mathcal{U}$ for which $x_0 \in \mathcal{S}$ and $f(x)$ is tangent to $\mathcal{S}$ for all $x \in \mathcal{S}$. The solution of the IVP $x' = f(x)$, $x(t_0) = x_0$ must stay on $\mathcal{S}$, so it follows that $T_+ = \infty$. A generalization of this example is the fact that an integral curve of a C^1 vector field on a compact manifold necessarily exists for all time.



Application of continuation theorem to linear systems

Consider the linear system $x'(t) = A(t)x(t) + b(t)$ for $a < t < b$ where $A(t) \in \mathbb{F}^{n\times n}$ and $b(t) \in \mathbb{F}^n$ are continuous on (a, b), with initial value $x(t_0) = x_0$ (where $t_0 \in (a, b)$). We allow the possibility that $a = -\infty$ and/or $b = \infty$. Let $\mathcal{D} = (a, b)\times\mathbb{F}^n$. Then $f(t, x) = A(t)x+b(t) \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on $\mathcal{D}$. Moreover, for c, d satisfying $a < c \le t_0 \le d < b$, f is uniformly Lipschitz continuous with respect to x on $[c, d] \times \mathbb{F}^n$ (we can take $L = \max_{c\le t\le d} |A(t)|$). The Picard global existence theorem implies there is a solution of the IVP on $[c, d]$, which is unique by the uniqueness theorem for locally Lipschitz f. This implies that $T_- = a$ and $T_+ = b$. We now give an alternate proof using the continuation theorem.

The idea is to prove an *a priori estimate* on the solution to show that $x(t)$ stays in a compact set in $\mathbb{F}^n$ for each compact subinterval of (a, b). Given d satisfying $t_0 < d < b$, let

$$M_d = \max_{t_0\le t\le d}(2|A(t)| + |b(t)|).$$

Suppose the solution $x(t)$ exists for $t_0 \le t \le d$ and let $u(t) = |x(t)|^2 = \langle x(t), x(t)\rangle$. Then by Cauchy-Schwarz,

$$\begin{aligned} u'(t) &= \langle x, x'\rangle + \langle x', x\rangle = 2\mathcal{R}e\langle x, x'\rangle \le 2|\langle x, x'\rangle| \le 2|x| \cdot |x'| \\ &= 2|x| \cdot |A(t)x + b(t)| \le 2|A(t)| \cdot |x|^2 + 2|b(t)| \cdot |x| \\ &\le 2|A(t)| \cdot |x|^2 + |b(t)|(|x|^2 + 1) \le M_d(|x|^2 + 1) = M_d(u + 1) \end{aligned}$$

(since $2|x| \le |x|^2 + 1$).

Gronwall's inequality (applied to $u' \le M_d u + M_d$ with $a(t) \equiv M_d$, $b(t) \equiv M_d$) implies that

$$u(t) \le u_0 e^{M_d(t-t_0)} + \int_{t_0}^{t} M_d e^{M_d(t-s)} ds = u_0 e^{M_d(t-t_0)} + e^{M_d(t-t_0)} - 1 \le R_d$$

for $t_0 \le t \le d$, where $u_0 = u(t_0)$ and $R_d = (u_0+1)e^{M_d(d-t_0)}-1$. So $|x(t)|^2 \le R_d$ for $t_0 \le t \le d$. If $T_+ < b$, it follows that $(t, x(t)) \in K$ for all $t < T_+$, where $K = [t_0, T_+] \times \{x : |x|^2 \le R_{T_+}\}$. This contradicts the continuation theorem. A similar argument shows that $T_- = a$.